

ORDERS OF AUTOMORPHISMS OF K3 SURFACES

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ABSTRACT. We determine all orders of automorphisms of complex K3 surfaces and of K3 surfaces in characteristic $p > 3$. The set $\text{Ord}_{\mathbb{C}}$ of all orders of automorphisms of finite order of complex K3 surfaces is given by

$$\text{Ord}_{\mathbb{C}} = \{N \mid N \text{ is a positive integer, } \phi(N) \leq 20\},$$

where ϕ is the Euler function, and the set Ord_p of all orders of automorphisms of finite order of K3 surfaces in characteristic p by

$$\text{Ord}_p = \begin{cases} \text{Ord}_{\mathbb{C}} & \text{if } p = 7 \text{ or } p > 19, \\ \text{Ord}_{\mathbb{C}} \setminus \{p, 2p\} & \text{if } p = 13, 17, 19, \\ \text{Ord}_{\mathbb{C}} \setminus \{44\} & \text{if } p = 11, \\ \text{Ord}_{\mathbb{C}} \setminus \{25, 50, 60\} & \text{if } p = 5. \end{cases}$$

In particular, 66 is the maximum possible order in each char. $p \neq 2, 3$.

1. INTRODUCTION

It is a natural and fundamental problem to determine all possible orders of automorphisms of K3 surfaces in any characteristic. Even in the case of complex K3 surfaces, this problem has been settled only for symplectic automorphisms and purely non-symplectic automorphisms (Nikulin [20], Kondō [15], Oguiso [22], Machida-Oguiso [18]). In this paper we solve the problem in all characteristics except 2 and 3. Let $\text{Ord}_{\mathbb{C}}$ and Ord_p be the sets of all orders of automorphisms of finite order respectively of complex K3 surfaces and K3 surfaces in characteristic p . Our main result is the following:

Main Theorem.

$$\text{Ord}_{\mathbb{C}} = \{N \mid N \text{ is a positive integer, } \phi(N) \leq 20\},$$

where ϕ is the Euler function, and

$$\text{Ord}_p = \begin{cases} \text{Ord}_{\mathbb{C}} & \text{if } p = 7 \text{ or } p > 19, \\ \text{Ord}_{\mathbb{C}} \setminus \{p, 2p\} & \text{if } p = 13, 17, 19, \\ \text{Ord}_{\mathbb{C}} \setminus \{44\} & \text{if } p = 11, \\ \text{Ord}_{\mathbb{C}} \setminus \{25, 50, 60\} & \text{if } p = 5. \end{cases}$$

In particular, 66 is the maximum possible order and $\text{Ord}_p \subset \text{Ord}_{\mathbb{C}}$ in each characteristic $p \neq 2, 3$. In characteristic $p = 2, 3$, we have $\phi(\text{ord}(g)) \leq 20$ for tame automorphisms g of finite order of K3 surfaces.

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The bound 66 is new even in characteristic 0. A previously known bound in characteristic 0 is $528 = 8 \times 66$ where 8 and 66 are the bounds respectively for symplectic automorphisms of finite order and purely non-symplectic automorphisms of finite order of complex K3 surfaces (Nikulin [20]). In characteristic $p > 0$ some bounds have been known only for special automorphisms, e.g., the bound 8 for tame symplectic automorphisms [7], the bound 11 for p -cyclic wild automorphisms [6], and the bound of Nygaard [21] for purely non-symplectic automorphisms of Artin-supersingular K3 surfaces: for an Artin-supersingular K3 surface X the image of $\text{Aut}(X)$ in $\text{GL}(H^0(X, \Omega_X^2))$ is cyclic of order dividing $p^{\sigma(X)} + 1$ where $\sigma(X)$ is the Artin invariant of X which may take values 1, 2, ..., 10.

In any characteristic, an automorphism of a K3 surface X is called *symplectic* if it acts trivially on the space $H^0(X, \Omega_X^2)$ of regular 2-forms, *non-symplectic* otherwise, and *purely non-symplectic* if it acts faithfully on the space $H^0(X, \Omega_X^2)$. A group of symplectic automorphisms is called *symplectic*. In characteristic $p > 0$, an automorphism of finite order will be called *tame* if its order is coprime to p and *wild* if divisible by p .

The list of finite groups which may act on a K3 surface is not yet known. In positive characteristic the list seems much longer than in characteristic 0. In fact, any finite group acting on a complex K3 surface is of order ≤ 3840 (Kondō [16]) and of order ≤ 960 if symplectic (Mukai [19]), while in characteristic $p = 5$ there is a K3 surface with a wild action of the simple group $\text{PSU}_3(5)$, whose order is 126,000 (cf. [7]), and in characteristic $p = 11$ a K3 surface with a wild action of the Mathieu group M_{22} , whose order is 443,520 (Dolgachev-Keum [8], Kondō [17]), and in infinitely many positive characteristics a K3 surface with a tame symplectic action of the Mathieu group M_{21} (Dolgachev-Keum [7]). Thus, it is natural to expect that in some characteristics p the set Ord_p would be bigger than the set $\text{Ord}_{\mathbb{C}}$. Our result shows that this never happens if $p > 3$ and may happen only in characteristic 2 or 3. It is interesting to note that elliptic curves share such properties (Remark 1.6).

The proof of Main Theorem will be divided into three cases: the tame case (Theorem 1.1), the complex case (Theorem 1.2) and the wild case (Theorem 1.5). We first determine all orders of tame automorphisms of K3 surfaces.

Theorem 1.1. *Let k be an algebraically closed field of characteristic $p > 3$. Let N be a positive integer not divisible by p . Then N is the order of an automorphism of a K3 surface X/k if and only if $\phi(N) \leq 20$.*

The if-part of Theorem 1.1 is proved in Section 3 by providing examples. Indeed, over $k = \mathbb{C}$ motivated by the Nikulin's fundamental work [20] Xiao [31], Machida and Oguiso [18] proved that a positive integer N is the order of a purely non-symplectic automorphism of a complex K3 surface if and only if $\phi(N) \leq 20$ and $N \neq 60$. They also provided examples of complex K3 surfaces with a purely non-symplectic automorphism of such an order N (see [18] Proposition 4, also [15] Section 7 and [22] Proposition 2.) In each

of their examples the K3 surface is defined over the integers and both the surface and the automorphism have a good reduction mod p as long as p is coprime to the order N and $p > 3$ (Proposition 3.5). Examples of K3 surfaces with an automorphism of order 60 in any characteristic $p \neq 2, 3, 5$ (including the zero characteristic) are given in Example 3.2. It can be shown (Keum [14]) that a K3 surface with a tame automorphism of order 60 is unique up to isomorphism, and the automorphism is unique up to conjugation, has non-symplectic order 12, and its 12th power is symplectic of order 5.

The only-if-part of Theorem 1.1 is proved in Section 4. Its proof depends on analyzing eigenvalues of the action on the l -adic cohomology $H_{\text{et}}^2(X, \mathbb{Q}_l)$, $l \neq p$. In positive characteristic, the holomorphic Lefschetz formula is not available and neither is the method using transcendental lattice. Our proof extends to the complex case, yielding a proof free from both, once we replace the l -adic cohomology by the singular integral cohomology. In other words, the positive characteristic case and the zero characteristic case can be handled in a unified fashion.

Theorem 1.2. *A positive integer N is the order of an automorphism of a complex K3 surface, projective or non-projective, if and only if $\phi(N) \leq 20$.*

Remark 1.3. By Deligne [4] any K3 surface in any positive characteristic lifts to characteristic 0. But automorphisms of K3 surfaces do not in general. J.-P. Serre [25] has proved the following result about lifting to characteristic 0: if X is a smooth projective variety over an algebraically closed field k of characteristic p with $H^2(X, \mathcal{O}_X) = H^2(X, \Theta_X) = 0$, where Θ_X is the tangent sheaf of X , and if $G \subset \text{Aut}(X)$ is a finite tame subgroup, then the pair (X, G) can be lifted to the ring $W(k)$ of Witt vectors. K3 surfaces do not satisfy the condition and the lifting theorem does not hold for them. In fact, two groups in the classification by Dolgachev and Keum of finite groups which may act tamely and symplectically on a K3 surface in positive characteristic have orders > 3840 (Theorem 5.2 [7]), hence cannot act (faithfully) on any complex K3 surface. In particular, there is a tame symplectic automorphism that does not lift to characteristic 0.

In Section 5, we prove the faithfulness of the representation of the automorphism group $\text{Aut}(X)$ on the l -adic cohomology $H_{\text{et}}^2(X, \mathbb{Q}_l)$, $l \neq p$, which is a result of Ogus (Corollary 2.5 [23]) when $p \neq 2$.

Theorem 1.4. *Let X be a K3 surface over an algebraically closed field k of characteristic $p > 0$. Then the representation*

$$\text{Aut}(X) \rightarrow \text{GL}(H_{\text{et}}^2(X, \mathbb{Q}_l)), \quad g \mapsto g^*,$$

is faithful for any prime $l \neq p$. In other words, an automorphism of X is determined by its action on $H_{\text{et}}^2(X, \mathbb{Q}_l)$.

In fact, Ogus proved under the condition that $p \neq 2$ the faithfulness of the representation on the crystalline cohomology $H_{\text{crys}}^2(X/W)$, and it is known

[10] that the characteristic polynomial of any automorphism has integer coefficients which do not depend on the choice of cohomology.

A wild automorphism of a K3 surface exists only in characteristic $p \leq 11$ (Theorem 2.1 [7]).

Theorem 1.5. *Let X be a K3 surface over an algebraically closed field k of characteristic $p = 11, 7$ or 5 . Let N be the order of an automorphism of finite order of X . Assume that the order N is divisible by p . Then*

$$N = pn$$

where $n = 1, 2, 3, 6$ if $p = 11$, $n = 1, 2, 3, 4, 6$ if $p = 7$, $n = 1, 2, 3, 4, 6, 8$ if $p = 5$. All these orders are supported by examples (Examples 7.5, 8.6, 9.5 and 9.11).

The proof of Theorem 1.5 is given in theorems 7.2, 8.2 and 9.3, and is based on the faithfulness (Theorem 1.4) and the results on wild p -cyclic actions on K3 surfaces [6], [7], [8].

In any characteristic $p \geq 0$ there are automorphisms of infinite order. For example, an elliptic K3 surface with Mordell-Weil rank positive always admits automorphisms of infinite order, namely, the automorphisms induced by the translation by a non-torsion section of the Mordell-Weil group of the Jacobian fibration. Such automorphisms are symplectic. Supersingular K3 surfaces (Ito [11]) and Kummer's quartic surfaces in characteristic $p \neq 2$ (Ueno [30]) also admit symplectic automorphisms of infinite order. Non-symplectic automorphisms of infinite order also exist in characteristic $p \neq 2$, e.g., on a generic Kummer's quartic surface the composition of an odd number of, more than one, projections [13], where a projection is the involution obtained by projecting the surface from a node onto \mathbb{P}^2 .

Remark 1.6. (1) It is well known that any group automorphism of an elliptic curve E is of finite order and acts faithfully on the first cohomology of E . The set of all possible orders of group automorphisms of elliptic curves does not depend on the characteristic and is given by

$$\{1, 2, 3, 4, 6\} = \{N \mid N \text{ is a positive integer, } \phi(N) \leq b_1 = 2\},$$

where b_1 is the first Betti number of an elliptic curve. But in characteristic 2 and 3, more groups may act on an elliptic curve. In fact, the quotient group $\text{Aut}(E)/\text{Aut}_0(E)$ where $\text{Aut}_0(E) \cong E$ is the identity component depends on the j -invariant of E and is isomorphic to the cyclic group \mathbb{Z}_2 , \mathbb{Z}_4 or \mathbb{Z}_6 in characteristic > 3 , to \mathbb{Z}_2 or $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ in characteristic 3, to \mathbb{Z}_2 or $Q_8 \rtimes \mathbb{Z}_3$ in characteristic 2 (see [28]).

(2) Unlike the elliptic curve case there are abelian varieties X with an automorphism g that acts faithfully on the first cohomology of X , but $\phi(\text{ord}(g)) > b_1(X)$. For example the n -fold product E^n of a general elliptic curve E with itself has $\text{Aut}(E^n)/\text{Aut}_0(E^n) = GL_n(\mathbb{Z})$ where $\text{Aut}_0(E^n) \cong E^n$ is the identity component. Take $n = 10$ and

$$g = (g_5, g_7) \in GL_4(\mathbb{Z}) \times GL_6(\mathbb{Z}) \subset GL_{10}(\mathbb{Z}),$$

where

$$g_q = \begin{pmatrix} 0 & 0 & \cdots & \cdots & -1 \\ 1 & 0 & \cdots & \cdots & -1 \\ 0 & 1 & \cdots & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \in GL_{q-1}(\mathbb{Z})$$

is an element of order q . Then $\text{ord}(g) = 35$ and $\phi(\text{ord}(g)) > b_1(E^{10}) = 20$.

Question. K3 surfaces in characteristic $\neq 2, 3$ and elliptic curves in any characteristic satisfy the inequality

$$\phi(\text{ord}(g)) \leq b_j(X)$$

for any automorphism g of the variety X of finite order that acts faithfully on the j -th cohomology of X . Do Calabi-Yau 3-folds so?

Notation

- $\text{NS}(X)$: the Néron-Severi group of a variety X ;
- $X^g = \text{Fix}(g)$: the fixed locus of an automorphism g of X .

For an automorphism g of a K3 surface X ,

- $\text{ord}(g) = m.n$: g is of order mn and the natural homomorphism $\langle g \rangle \rightarrow \text{GL}(H^0(X, \Omega_X^2)) \cong k^*$ has kernel of order m and image of order n ;
- $[g^*] = [\lambda_1, \dots, \lambda_{22}]$: the list of the eigenvalues of $g^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$.
- ζ_a : a primitive a -th root of unity in $\overline{\mathbb{Q}_l}$;
- $[\zeta_a : \phi(a)] \subset [g^*]$: all primitive a -th roots of unity appear in $[g^*]$ where $\phi(a)$ indicates the number of them.
- $[\lambda.r] \subset [g^*]$: λ repeats r times in $[g^*]$.
- $[(\zeta_a : \phi(a)).r] \subset [g^*]$: the list $\zeta_a : \phi(a)$ repeats r times in $[g^*]$.
- ϕ : the Euler function

2. PRELIMINARIES

Proposition 2.1. *Let X be a projective variety over an algebraically closed field k of characteristic $p > 0$. Let g be an automorphism of X . Let $l \neq p$. Then the following hold true.*

- (1) (3.7.3 [10]) *The characteristic polynomial of $g^*|H_{\text{et}}^j(X, \mathbb{Q}_l)$ has integer coefficients for each j .*
- (2) *If g is of finite order, then g has an invariant ample divisor, hence $g^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$ has 1 as an eigenvalue.*
- (3) *If X is a K3 surface and g^* acts trivially on $H_{\text{et}}^2(X, \mathbb{Q}_l)$, then g^* acts trivially on the space of regular 2-forms $H^0(X, \Omega_X^2)$.*

- (4) If X is a K3 surface, g is tame and $g^*|H^0(X, \Omega_X^2)$ has $\zeta_n \in k$ as an eigenvalue, then $g^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$ has $\zeta_n \in \overline{\mathbb{Q}_l}$ as an eigenvalue.

Proof. (2) For any ample divisor D the sum $\sum g^i(D)$ is g -invariant.

(3) The characteristic polynomial of g^* has integer coefficients, which do not depend on the choice of a cohomology theory. Consider the representation of $\langle g \rangle$ on the second crystalline cohomology $H_{\text{crys}}^2(X/W)$ where $W = W(k)$ is the ring of Witt vectors. See [10] for the crystalline cohomology. The quotient module

$$H_{\text{crys}}^2(X/W)/pH_{\text{crys}}^2(X/W)$$

is a finite dimensional k -vector space isomorphic to the algebraic de Rham cohomology $H_{\text{DR}}^2(X)$. Thus the representation of $\langle g \rangle$ on $H_{\text{DR}}^2(X)$ is also trivial. It is known that the Hodge to de Rham spectral sequence

$$E_1^{t,s} := H^s(X, \Omega_X^t) \Rightarrow H_{\text{DR}}^*(X)$$

degenerates at E_1 , giving the Hodge filtration on $H_{\text{DR}}^*(X)$. In particular g^* acts trivially on the space of regular 2-forms $H^0(X, \Omega_X^2)$. \square

Recall that for a nonsingular projective variety Z in characteristic $p > 0$, there is an exact sequence of \mathbb{Q}_l -vector spaces

$$(2.1) \quad 0 \rightarrow \text{NS}(Z) \otimes \mathbb{Q}_l \rightarrow H_{\text{et}}^2(Z, \mathbb{Q}_l) \rightarrow T_l^2(Z) \rightarrow 0$$

where $T_l^2(Z) = T_l(\text{Br}(Z))$ in the standard notation in the theory of étale cohomology. The Brauer group $\text{Br}(Z)$ is known to be a birational invariant, and it is trivial when Z is a rational variety. In fact, one can show that

$$\text{NS}(Z) \otimes \mathbb{Q}_l = \text{Ker}(H_{\text{et}}^2(Z, \mathbb{Q}_l) \rightarrow H^2(k(Z), \mathbb{Q}_l));$$

$$T_l^2(Z) = \text{Im}(H_{\text{et}}^2(Z, \mathbb{Q}_l) \rightarrow H^2(k(Z), \mathbb{Q}_l)).$$

Here $H^2(k(Z), \mathbb{Q}_l) = \varinjlim_U H^2(U, \mathbb{Q}_l)$, where U runs through the set of open subsets of Z (see [26]). It is known that the dimension of all \mathbb{Q}_l -spaces from above do not depend on l prime to the characteristic p .

Proposition 2.2. *In the situation as above, let g be an automorphism of Z of finite order. Assume $l \neq p$. Then the following assertions are true.*

- (1) Both traces of g^* on $\text{NS}(Z)$ and on $T_l^2(Z)$ are integers.
- (2) $\text{rank NS}(Z)^g = \text{rank NS}(Z/\langle g \rangle)$.
- (3) $\dim H_{\text{et}}^2(Z, \mathbb{Q}_l)^g = \text{rank NS}(Z)^g + \dim T_l^2(Z)^g$.
- (4) If the minimal resolution Y of $Z/\langle g \rangle$ has $T_l^2(Y) = 0$, then

$$\dim H_{\text{et}}^2(Z, \mathbb{Q}_l)^g = \text{rank NS}(Z)^g.$$

The condition of (4) is satisfied if $Z/\langle g \rangle$ is rational or is birational to an Enriques surface.

Proof. (4) By Proposition 5 [26], $T_l^2(Z)^g \cong T_l^2(Y) = 0$. Hence the result follows from (3). \square

We use the following result of Deligne and Lusztig, which is an extension to the case of wild automorphisms of the Lefschetz fixed point formula.

Proposition 2.3. (*Theorem 3.2 [5]*) *Let X be a scheme, separated and of finite type over an algebraically closed field k of characteristic $p > 0$ and let g be an automorphism of finite order of X . We decompose g as $g = su$ where s and u are powers of g respectively of order prime to p and a power of p . Then*

$$\sum_j (-1)^j \text{Tr}(g^* | H_c^j(X, \mathbb{Q}_l)) = \sum_j (-1)^j \text{Tr}(u^* | H_c^j(X^s, \mathbb{Q}_l))$$

where the cohomology is the l -adic rational cohomology with compact support.

Proposition 2.4. (*Lefschetz fixed point formula*) *Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$ and let g be a tame automorphism of X . Then $X^g = \text{Fix}(g)$ is smooth and*

$$e(X^g) = \sum_j (-1)^j \text{Tr}(g^* | H_{\text{et}}^j(X, \mathbb{Q}_l)).$$

Proof. This follows from Proposition 2.3. The tame case is just the case with $s = g$ and $u = 1_X$. Note that $\sum_j (-1)^j \text{Tr}(1^* | H_{\text{et}}^j(X^g, \mathbb{Q}_l)) = e(X^g)$. \square

Proposition 2.5. (*Theorem 3.3 and Proposition 4.1 [7]*) *A tame symplectic automorphism h of a K3 surface has finitely many fixed points, the number of fixed points $f(h)$ depends only on the order of h and the list of possible pairs $(\text{ord}(h), f(h))$ is the same as in the complex case:*

$$(\text{ord}(h), f(h)) = (2, 8), (3, 6), (4, 4), (5, 4), (6, 2), (7, 3), (8, 2).$$

Lemma 2.6. *Let h be a tame symplectic automorphism of a K3 surface X . Then $h^* | H_{\text{et}}^2(X, \mathbb{Q}_l)$ has eigenvalues*

$$\begin{aligned} \text{ord}(h) = 2 & : [h^*] = [1, 1.13, -1.8] \\ \text{ord}(h) = 3 & : [h^*] = [1, 1.9, (\zeta_3 : 2).6] \\ \text{ord}(h) = 4 & : [h^*] = [1, 1.7, (\zeta_4 : 2).4, -1.6] \\ \text{ord}(h) = 5 & : [h^*] = [1, 1.5, (\zeta_5 : 4).4] \\ \text{ord}(h) = 6 & : [h^*] = [1, 1.5, (\zeta_3 : 2).4, (\zeta_6 : 2).2, -1.4] \\ \text{ord}(h) = 7 & : [h^*] = [1, 1.3, (\zeta_7 : 6).3] \\ \text{ord}(h) = 8 & : [h^*] = [1, 1.3, (\zeta_8 : 4).2, (\zeta_4 : 2).3, -1.4] \end{aligned}$$

where the first eigenvalue corresponds to an invariant ample divisor.

Proof. The result follows from Proposition 2.4 and Proposition 2.5. \square

Lemma 2.7. *Let X be a K3 surface in characteristic $p \neq 2$. If h is a non-symplectic automorphism of order 4 of X such that h^2 is symplectic, then X^h is finite and $0 \leq e(X^h) \leq 8$.*

Proof. Note that $\text{Fix}(h) \subset \text{Fix}(h^2)$ and the latter consists of 8 points. \square

The following lemma will be used in analyzing wild automorphisms.

Lemma 2.8. *Let X be a K3 surface in characteristic $p \neq 2, 3$. Assume that h is an automorphism of order 2 with $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^h = 2$. Then h is non-symplectic and has a h -invariant elliptic fibration $\psi : X \rightarrow \mathbb{P}^1$ with 12 cuspidal fibers, and X^h consists of either a curve of genus 9 which is a 4-section of ψ passing through each cusp with multiplicity 3 or a section and a curve of genus 10 which is a 3-section passing through each cusp with multiplicity 3.*

Proof. Since $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^h = 2$, the eigenvalues of $h^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$ must be

$$[h^*] = [1.2, -1.20],$$

so

$$\sum (-1)^j \text{Tr}(h^*|H_{\text{et}}^j(X, \mathbb{Q}_l)) = -16.$$

By Lemma 2.6, h is non-symplectic, thus X^h is a disjoint union of smooth curves and the quotient $X/\langle h \rangle$ is a nonsingular rational surface. Note that $e(X^h) = -16$, hence X^h is non-empty and has at most 2 components. Thus X^h is either a curve of genus 9 or two curves of genus 0 and 10. By Proposition 2.2, $X/\langle h \rangle$ has Picard number 2, hence is isomorphic to a rational ruled surface \mathbb{F}_e . Let S_0 be the section of \mathbb{F}_e with $S_0^2 = -e$, and F be a fibre. If X^h is a curve C_9 of genus 9, then its image C'_9 in \mathbb{F}_e satisfies $C_9'^2 = 32$ and $C'_9 K = -16$, hence $C'_9 \equiv 4S_0 + (4 + 2e)F$. Assume that X^h consists of two curves C_0 and C_{10} of genus 0 and 10. Then $C_0'^2 = -4$, hence $C'_0 = S_0$ and $e = 4$. Then it is easy to see that $C_{10}' \equiv 3(S_0 + 4F)$. The pullback of the ruling on \mathbb{F}_e gives a h -invariant elliptic fibration $\psi : X \rightarrow \mathbb{P}^1$. It must have 12 cuspidal fibres and X^h passes through each cusp with multiplicity 3. \square

We will use frequently the Weyl theorem of the following form.

Lemma 2.9. *Let V be a finite dimensional vector space over a field of characteristic 0. Let $g \in GL(V)$ be a linear automorphism of finite order. Assume that the characteristic polynomial of g has integer coefficients. If for some positive integer m a primitive m -th root of unity appears with multiplicity r as an eigenvalue of g , then so does each of its conjugates.*

Proof. Let $n = \text{ord}(g)$. Every eigenvalue of g is an n -th root of unity, not necessarily primitive. If a primitive m -th root of unity appears as an eigenvalue of g , then its cyclotomic polynomial must divide the characteristic polynomial of g . \square

The following easy lemmas also will be used frequently.

Lemma 2.10. *Let S be a set and $\text{Aut}(S)$ be the group of bijections of S . For any $g \in \text{Aut}(S)$ and positive integers a and b ,*

- (1) $\text{Fix}(g) \subset \text{Fix}(g^a)$;
- (2) $\text{Fix}(g^a) \cap \text{Fix}(g^b) = \text{Fix}(g^d)$ where $d = \gcd(a, b)$;
- (3) $\text{Fix}(g) = \text{Fix}(g^a)$ if $\text{ord}(g)$ is finite and prime to a .

Lemma 2.11. *Let g be an automorphism of \mathbb{P}^1 over an algebraically closed field k of characteristic $p > 0$.*

- (1) *If $g^{pm} = 1$, $g^m \neq 1$ and m is coprime to p , then $g^p = 1$.*
- (2) *If $g^{p^2} = 1$, then $g^p = 1$.*

Proof. easily follows from the Jordan canonical form. \square

Lemma 2.12. *Let $R(n)$ be the sum of all primitive n -th root of unity in $\overline{\mathbb{Q}}$ or in $\overline{\mathbb{Q}_l}$. Then*

$$R(n) = \begin{cases} 0 & \text{if } n \text{ has a square factor,} \\ (-1)^t & \text{if } n \text{ is a product of } t \text{ distinct primes.} \end{cases}$$

Proof. Let $n = q_1^{r_1} q_2^{r_2} \cdots q_t^{r_t}$ be the prime factorization. It is easy to see that a primitive n -th root of unity can be factorized uniquely into a product of a primitive $q_1^{r_1}$ -th root, ... , a primitive $q_t^{r_t}$ -th root of unity, thus

$$R(n) = R(q_1^{r_1}) R(q_2^{r_2}) \cdots R(q_t^{r_t}).$$

For any prime q , $R(q) = -1$ and $R(q^r) = 0$ if $r > 1$. \square

3. EXAMPLES OF AUTOMORPHISMS

In this section we will prove the if-part of Theorem 1.1 and Theorem 1.2 by providing examples. See Propositions 3.4 and 3.5.

Let X be a complex K3 surface. The transcendental lattice T_X of X is by definition the orthogonal complement of the Néron-Severi group in the cohomology lattice $H^2(X, \mathbb{Z})$. If X is not projective, then the image of the homomorphism

$$\text{Aut}(X) \rightarrow \text{GL}(H^0(X, \Omega_X^2)) \cong \mathbb{C}^*$$

is either trivial or an infinite cyclic group (cf. Ueno [29]). Thus every automorphism of finite order of a non-projective K3 surface is symplectic, hence of order ≤ 8 (Nikulin [20]).

Let X be a projective complex K3 surface and let g be an automorphism of non-symplectic order n , i.e., its image in $\text{GL}(H^0(X, \Omega_X^2))$ is of order n . We may regard the transcendental lattice T_X as a $\mathbb{Z}[\langle g \rangle]$ -module via the natural action of $\langle g \rangle$ on T_X . Since g has non-symplectic order n , T_X becomes a free $\mathbb{Z}[\langle g \rangle]/\langle \Phi_n(g) \rangle$ -module where $\Phi_n(x) \in \mathbb{Z}[x]$ is the n -th cyclotomic polynomial [20], thus T_X can be viewed as a free $\mathbb{Z}[\zeta_n]$ -module via the isomorphism

$$\mathbb{Z}[\langle g \rangle]/\langle \Phi_n(g) \rangle \cong \mathbb{Z}[\zeta_n]$$

where ζ_n is a primitive n -th root of unity. In particular $\phi(n)$ divides $\text{rank } T_X$ where ϕ is the Euler function. Since X is projective, $\text{rank } T_X \leq 21$ and hence

$$\phi(n) \leq 21.$$

Motivated by this, Kondō [15], Xiao [31], Machida and Oguiso [18] studied purely non-symplectic automorphisms of complex K3 surfaces and proved that a positive integer n is the order of a purely non-symplectic automorphism of a complex K3 surface if and only if $\phi(n) \leq 20$ and $n \neq 60$. They

also provided examples of complex K3 surfaces with a purely non-symplectic automorphism of such an order n .

For convenience, we list all integers n with $\phi(n) \leq 21$ in Table 1. There is no integer n with $\phi(n) = 21$.

TABLE 1. The list of all integers n with $\phi(n) \leq 21$

$\phi(n)$	20	18	16	12	10	8	6	4	2	1
n	66	54	60	42	22	30	18	12	6	2
	50	38	48	36	11	24	14	10	4	1
	44	27	40	28		20	9	8	3	
	33	19	34	26		16	7	5		
	25		32	21		15				
			17	13						

From Table 1 we see that a positive integer n satisfying $n \neq 60$ and $\phi(n) \leq 20$ is a divisor of an element of the set

$$\mathcal{M}_{pns} := \{66, 50, 44, 54, 38, 48, 40, 34, 32, 42, 36, 28, 26, 30\}.$$

Example 3.1. (Proposition 4 [18], also Section 7 [15], Proposition 2 [22]) For each $n \in \mathcal{M}_{pns}$, there is a K3 surface X_n with a purely non-symplectic automorphism g_n of order n . The surface X_n is defined by the indicated Weierstrass equation for $n \neq 50, 40$. The surface $X_{50} \subset \mathbb{P}(1, 1, 1, 3)$ is defined as a double plane branched along a smooth sextic curve and the surface $X_{40} \subset \mathbb{P}(1, 1, 1, 3)$ as (the minimal resolution of) a double plane branched along the union of a line and a smooth quintic curve.

- (1) $X_{66} : y^2 = x^3 + t(t^{11} - 1), \quad g_{66}(t, x, y) = (\zeta_{66}^{54}t, \zeta_{66}^{40}x, \zeta_{66}^{27}y);$
- (2) $X_{50} : w^2 = x^6 + xy^5 + yz^5, \quad g_{50}(x, y, z, w) = (x, \zeta_{50}^{40}y, \zeta_{50}^2z, \zeta_{50}^{25}w);$
- (3) $X_{44} : y^2 = x^3 + x + t^{11}, \quad g_{44}(t, x, y) = (\zeta_{44}^{34}t, \zeta_{44}^{22}x, \zeta_{44}^{11}y);$
- (4) $X_{54} : y^2 = x^3 + t(t^9 - 1), \quad g_{54}(t, x, y) = (\zeta_{54}^{12}t, \zeta_{54}^4x, \zeta_{54}^{33}y);$
- (5) $X_{38} : y^2 = x^3 + t^7x + t, \quad g_{38}(t, x, y) = (\zeta_{38}^4t, \zeta_{38}^{14}x, \zeta_{38}^{21}y);$
- (6) $X_{48} : y^2 = x^3 + t(t^8 - 1), \quad g_{48}(t, x, y) = (\zeta_{48}^6t, \zeta_{48}^2x, \zeta_{48}^3y);$
- (7) $X_{40} : w^2 = x(x^4z + y^5 - z^5), \quad g_{40}(x, y, z, w) = (x, \zeta_{40}^2y, \zeta_{40}^{10}z, \zeta_{40}^5w);$
- (8) $X_{34} : y^2 = x^3 + t^7x + t^2, \quad g_{34}(t, x, y) = (\zeta_{34}^4t, \zeta_{34}^{14}x, \zeta_{34}^{21}y);$
- (9) $X_{32} : y^2 = x^3 + t^2x + t^{11}, \quad g_{32}(t, x, y) = (\zeta_{32}^2t, \zeta_{32}^{18}x, \zeta_{32}^{27}y);$
- (10) $X_{42} : y^2 = x^3 + t^5(t^7 - 1), \quad g_{42}(t, x, y) = (\zeta_{42}^{18}t, \zeta_{42}^2x, \zeta_{42}^3y);$

- (11) $X_{36} : y^2 = x^3 + t^5(t^6 - 1), \quad g_{36}(t, x, y) = (\zeta_{36}^{30}t, \zeta_{36}^2x, \zeta_{36}^3y);$
- (12) $X_{28} : y^2 = x^3 + x + t^7, \quad g_{28}(t, x, y) = (\zeta_{28}^2t, \zeta_{28}^{14}x, \zeta_{28}^7y);$
- (13) $X_{26} : y^2 = x^3 + t^5x + t, \quad g_{26}(t, x, y) = (\zeta_{26}^4t, \zeta_{26}^{10}x, \zeta_{26}^{15}y);$
- (14) $X_{30} : y^2 = x^3 + (t^{10} - 1), \quad g_{30}(t, x, y) = (\zeta_{30}^3t, \zeta_{30}^{10}x, y).$

We give an example of a K3 surface with an order 60 automorphism.

Example 3.2. In char $p \neq 2, 3, 5$, there is a K3 surface with an automorphism of order $60 = 5 \cdot 12$

$$X_{60} : y^2 + x^3 + t^{11} - t = 0, \quad g_{60}(t, x, y) = (\zeta_{60}^6t, \zeta_{60}^2x, \zeta_{60}^3y).$$

The surface X has 12 type II -fibres at $t = \infty, t^{11} - t = 0$.

Remark 3.3. In [14] it has been shown that in characteristic $p \geq 0, p \neq 2, 3, 5$, a K3 surface with an automorphism of order 60 is unique up to isomorphism and the automorphism is unique up to conjugation and must have order 5.12.

A positive integer n with $\phi(n) \leq 20$ is a divisor of an element of the set

$$\mathcal{M} := (\mathcal{M}_{pns} \setminus \{30\}) \cup \{60\}.$$

Proposition 3.4. *For any positive integer N satisfying $\phi(N) \leq 20$, there is a complex K3 surface with an automorphism of order N .*

Proposition 3.5. *For any positive integer N satisfying $\phi(N) \leq 20$ and any prime $p > 3$ coprime to N , there is a K3 surface in characteristic p with an automorphism of order N .*

Proof. In each of the above examples the K3 surface X_n is defined over the integers. Fix a prime $p > 3$. Then a positive integer n with $\phi(n) \leq 20$, not divisible by p , is a divisor of an element of the set \mathcal{M}_p , where

$$\mathcal{M}_p := \begin{cases} \mathcal{M} & \text{if } p > 19, \\ \mathcal{M} \setminus \{2p\} & \text{if } p = 13, 17, 19, \\ \mathcal{M} \setminus \{66, 44\} & \text{if } p = 11, \\ \mathcal{M} \setminus \{42, 28\} & \text{if } p = 7, \\ \mathcal{M} \setminus \{50, 40, 60\} & \text{if } p = 5. \end{cases}$$

It is easy to check that for each $n \in \mathcal{M}_p$ the pair (X_n, g_n) has a good reduction mod p , i.e., the surface X_n defines a K3 surface in characteristic p and g_n defines an automorphism of order n . \square

4. THE TAME CASE

In this section we will prove the following:

Proposition 4.1. *Let k be an algebraically closed field of characteristic $p > 0$. Let N be a positive integer not divisible by p . If N is the order of an automorphism of a K3 surface X/k , then $\phi(N) \leq 20$ where ϕ is the Euler function.*

Let g be a tame automorphism of order N of a K3 surface, i.e., the order N is prime to the characteristic p . Assume that

$$\text{ord}(g) = N = m \cdot n$$

i.e., g is of order $N = mn$ and its action on the space $H^0(X, \Omega_X^2)$ has order n . Then g^n is symplectic of order m , hence by Proposition 2.5,

$$m \leq 8.$$

By Proposition 2.1(4), $g^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$ has $\zeta_n \in \overline{\mathbb{Q}_l}$ as an eigenvalue. The second cohomology space has dimension 22 and g^* has 1 as an eigenvalue corresponding to an invariant ample divisor, so by Proposition 2.1 and Lemma 2.9,

$$\phi(n) \leq 21.$$

See Table 1 for all integers n with $\phi(n) \leq 21$.

We will prove $\phi(N) \leq 20$ in the following lemmas 4.2—4.8.

Lemma 4.2. *If $\phi(n) > 13$, then $m = 1$.*

Proof. The primitive n -th root ζ_n is an eigenvalue of $g^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$, hence

$$[1, \zeta_n : \phi(n)] \subset [g^*]$$

where the first eigenvalue $1 \in [g^*]$ corresponds to an invariant ample divisor (Proposition 2.1), $\zeta_n : \phi(n)$ means all primitive n -th roots of unity and $\phi(n)$ indicates the number of them. Thus we infer that g^n is symplectic of order m such that

$$[1, 1.\phi(n)] \subset [g^{n*}].$$

If $\phi(n) > 13$, then $m = 1$ by Lemma 2.6. □

Lemma 4.3. *Assume that $\phi(n) = 12$. Then $m = 1$ if $n = 42, 36, 28, 26$, and $m \leq 2$ if $n = 21, 13$.*

Proof. Since $[1, \zeta_n : \phi(n)] \subset [g^*]$, $[1, 1.12] \subset [g^{n*}]$. Since g^n is symplectic of order m , $m \leq 2$ by Lemma 2.6.

Assume $m = 2$. Then g^n is symplectic of order 2 and by Lemma 2.6

$$[g^{n*}] = [1, 1.12, 1, -1.8].$$

Assume that $n = 2n'$. Then $g^{n'}$ is a non-symplectic automorphism of order 4 such that its square is symplectic. Since $\zeta_n^{n'} = -1$, we infer that

$$[g^{n'*}] = [1, -1.12, \pm 1, (\zeta_4 : 2).4].$$

In any case, the Lefschetz fixed point formula (Proposition 2.4) yields

$$e(\text{Fix}(g^{n'})) < 0,$$

contradicting Lemma 2.7. □

Lemma 4.4. *Assume that $\phi(n) = 10$. Then $m = 1$ if $n = 22$ and $m \leq 2$ if $n = 11$.*

Proof. Since $[1, \zeta_n : \phi(n)] \subset [g^*]$, $[1, 1.10] \subset [g^{n*}]$. Since g^n is symplectic of order m , $m \leq 2$ by Lemma 2.6.

Assume that $m = 2$ and $n = 22$. Then g^{11} is non-symplectic of order 4 with a symplectic square. Since

$$[g^{22*}] = [1, 1.10, -1.8, 1.3],$$

we infer that

$$[g^{11*}] = [1, -1.10, (\zeta_4 : 2).4, \pm 1, \pm 1, \pm 1].$$

In any case, $\text{Tr}(g^{11*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) < -2$, contradicting Lemma 2.7. \square

Lemma 4.5. *Assume $\phi(n) = 8$.*

- (1) *If $n = 30, 24, 20$ or 16 , then $m = 1$.*
- (2) *If $n = 15$, then $m \leq 2$.*

Proof. Since $[1, \zeta_n : \phi(n)] \subset [g^*]$, $[1, 1.8] \subset [g^{n*}]$. Since g^n is symplectic of order m , $m \leq 3$ by Lemma 2.6.

Claim: $\text{ord}(g) \neq 2.16$.

On the contrary, suppose that $\text{ord}(g) = 2.16$. Then $-1 \in [g^{16*}]$. Thus $\zeta_{32} \in [g^*]$, then $[-1.\phi(32)] = [-1.16] \subset [g^{16*}] = [1, 1.13, -1.8]$.

Claim: $\text{ord}(g) \neq 3.16$.

Suppose that $\text{ord}(g) = 3.16$. Then $[g^{16*}] = [1, 1.9, (\zeta_3 : 2).6]$. We infer that

$$[g^*] = [1, \zeta_{16} : 8, \pm 1, \eta_1, \dots, \eta_{12}]$$

where η_1, \dots, η_{12} is a combination of $\zeta_3 : 2$, $\zeta_6 : 2$, $\zeta_{12} : 4$ and $\zeta_{24} : 8$. In any case, $[g^{8*}] = [1, -1.8, 1, (\zeta_3 : 2).6]$ and

$$\text{Tr}(g^{8*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = -12.$$

On the other hand, $\text{Fix}(g^8)$ is contained in the finite set $\text{Fix}(g^{16})$, hence

$$\text{Tr}(g^{8*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = e(\text{Fix}(g^8)) - 2 \geq -2.$$

Claim: $\text{ord}(g) \neq 3.15$.

Suppose that $\text{ord}(g) = 3.15$. Then $[g^{15*}] = [1, 1.9, (\zeta_3 : 2).6]$. Since $\phi(45) > 12$, $\zeta_{45} \notin [g^*]$ and we infer that

$$[g^*] = [1, \zeta_{15} : 8, 1, (\zeta_9 : 6).2].$$

Thus $[g^{3*}] = [1, (\zeta_5 : 4).2, 1, (\zeta_3 : 2).6]$, hence

$$\text{Tr}(g^{3*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = -6.$$

On the other hand, $\text{Fix}(g^3)$ is contained in $\text{Fix}(g^{15})$, so is finite.

Claim: $\text{ord}(g) \neq 3.30$.

Suppose that $\text{ord}(g) = 3.30$. Then $\text{ord}(g^2) = 3.15$.

Claim: $\text{ord}(g) \neq 3.24$.

Suppose that $\text{ord}(g) = 3.24$. Then $[g^{24*}] = [1, 1.8, 1, (\zeta_3 : 2).6]$, thus

$$[g^*] = [1, \zeta_{24} : 8, \pm 1, \eta_1, \dots, \eta_{12}]$$

where $[\eta_1, \dots, \eta_{12}]$ is a combination of $\zeta_9 : 6$, $\zeta_{18} : 6$ and $\zeta_{36} : 12$. The 4th power of ζ_9 , ζ_{18} , ζ_{36} is a 9th root of unity, so we infer that

$$[g^{4*}] = [1, (\zeta_6 : 2).4, 1, (\zeta_9 : 6).2].$$

Thus

$$\text{Tr}(g^{4*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 6 > \text{Tr}(g^{24*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 4.$$

But $\text{Fix}(g^4) \subset \text{Fix}(g^{24})$ a finite set, so the inequality is impossible.

Claim: $\text{ord}(g) \neq 2.24$.

Suppose that $\text{ord}(g) = 2.24$. Then $[g^{24*}] = [1, 1.8, 1.5, -1.8]$. We infer that

$$[g^*] = [1, \zeta_{24} : 8, \eta_1, \dots, \eta_5, \zeta_{16} : 8]$$

where η_1, \dots, η_5 is a combination of 1, -1 , $\zeta_3 : 2$, $\zeta_4 : 2$, $\zeta_6 : 2$, $\zeta_8 : 4$ and $\zeta_{12} : 4$. In any case, $\sum_j \eta_j^8 \leq 5$ and

$$\text{Tr}(g^{8*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 - 4 + \sum \eta_j^8 - 8 \leq -6.$$

On the other hand, $\text{Fix}(g^8)$ is contained in $\text{Fix}(g^{24})$, hence is finite.

Claim: $\text{ord}(g) \neq 3.20$.

Suppose that $\text{ord}(g) = 3.20$. Then $[g^{20*}] = [1, 1.8, 1, (\zeta_3 : 2).6]$, thus

$$[g^*] = [1, \zeta_{20} : 8, \pm 1, \eta_1, \dots, \eta_{12}]$$

where η_1, \dots, η_{12} is a combination of $\zeta_3 : 2$, $\zeta_6 : 2$, $\zeta_{12} : 4$, $\zeta_{15} : 8$, $\zeta_{30} : 8$. We claim that $[\eta_1, \dots, \eta_{12}] = [(\zeta_{12} : 4).3]$. Otherwise, $\sum_j \eta_j^{10} \leq 2$ and

$$\text{Tr}(g^{10*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 - 8 + 1 + \sum \eta_j^{10} \leq -4,$$

hence $e(\text{Fix}(g^{10})) < 0$, but $\text{Fix}(g^{10})$ is contained in $\text{Fix}(g^{20})$. This proves the claim and we have

$$[g^*] = [1, \zeta_{20} : 8, \pm 1, (\zeta_{12} : 4).3].$$

Then

$$[g^{4*}] = [1, (\zeta_5 : 4).2, 1, (\zeta_3 : 2).6],$$

thus $\text{Tr}(g^{4*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = -6$ contradicting $\text{Fix}(g^4) \subset \text{Fix}(g^{20})$.

Claim: $\text{ord}(g) \neq 2.20$.

Suppose that $\text{ord}(g) = 2.20$. Then $[g^{20*}] = [1, 1.8, 1.5, -1.8]$, thus

$$[g^*] = [1, \zeta_{20} : 8, \eta_1, \dots, \eta_5, (\zeta_8 : 4).2]$$

where η_1, \dots, η_5 is a combination of 1, -1 , $\zeta_4 : 2$, $\zeta_5 : 4$ and $\zeta_{10} : 4$. In any case, $\sum_j \eta_j^4 \leq 5$ and $\text{Tr}(g^{4*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 - 2 + \sum \eta_j^4 - 8 \leq -4$, contradicting $\text{Fix}(g^4) \subset \text{Fix}(g^{20})$.

Claim: $\text{ord}(g) \neq 2.30$.

Suppose that $\text{ord}(g) = 2.30$. Then $[g^{30*}] = [1, 1.8, 1.5, -1.8]$. We infer that

$$[g^*] = [1, \zeta_{30} : 8, \eta_1, \dots, \eta_5, \tau_1, \dots, \tau_8]$$

where $[\eta_1, \dots, \eta_5]$ is a combination of 1, -1 , $\zeta_3 : 2$, $\zeta_6 : 2$, $\zeta_5 : 4$, $\zeta_{10} : 4$, and $[\tau_1, \dots, \tau_8]$ is a combination of $\zeta_4 : 2$, $\zeta_{12} : 4$ and $\zeta_{20} : 8$. In any case, we see that

$$[g^{15*}] = [1, -1.8, \eta_1^{15}, \dots, \eta_5^{15}, (\zeta_4 : 2).4].$$

Since $\eta_j^{15} = \pm 1$, we see that $\sum \eta_j^{15} \leq 5$ and

$$-2 \leq \text{Tr}(g^{15*} | H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 - 8 + \sum \eta_j^{15} + 0 \leq -2.$$

Thus $\eta_1^{15} = \dots = \eta_5^{15} = 1$. This occurs iff $[\eta_1, \dots, \eta_5]$ is a combination of 1, $\zeta_3 : 2$ and $\zeta_5 : 4$.

If $[\eta_1, \dots, \eta_5]$ is a combination of 1 and $\zeta_3 : 2$, then

$$\text{Tr}(g^{3*} | H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 + 2 + 5 + 0 = 8 > \text{Tr}(g^{30*} | H_{\text{et}}^2(X, \mathbb{Q}_l)) = 6,$$

a contradiction. Here we use Lemma 2.12 for the sum of the conjugates of ζ_{10} and the sum of the conjugates of ζ_{20} .

If $[\eta_1, \dots, \eta_5] = [\zeta_5 : 4, 1]$, then

$$\text{Tr}(g^{5*} | H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 + 4 + 5 + 0 = 10 > \text{Tr}(g^{30*} | H_{\text{et}}^2(X, \mathbb{Q}_l)) = 6,$$

a contradiction. Here we use that the sum of the conjugates of ζ_{12} is 0. \square

Lemma 4.6. *Assume that $\phi(n) = 6$.*

- (1) *If $n = 18$, then $m = 1$.*
- (2) *If $n = 9$, then $m \leq 2$.*
- (3) *If $n = 14$, then $m = 1$ or 3.*
- (4) *If $n = 7$, then $m \leq 3$.*

Proof. We see that g^n is symplectic of order m with $[1, 1.6] \subset [g^{n*}]$, thus $m \leq 4$ by Lemma 2.6.

Assume that $m = 3$. Then $[g^{n*}] = [1, 1.9, (\zeta_3 : 2).6]$. If $n = 9$ or 18, then ζ_{27} or $\zeta_{54} \in [g^*]$, but $\phi(27) = \phi(54) = 18 > 12$.

It remains to show that 2.18, 4.9, 4.7 and 2.14 do not occur.

Claim: $\text{ord}(g) \neq 2.18$.

Suppose that $\text{ord}(g) = 2.18$. Then $[g^{18*}] = [1, 1.6, 1.7, -1.8]$, thus

$$[g^*] = [1, \zeta_{18} : 6, \eta_1, \dots, \eta_7, \tau_1, \dots, \tau_8]$$

where $[\eta_1, \dots, \eta_7]$ is a combination of 1, -1 , $\zeta_3 : 2$, $\zeta_6 : 2$, $\zeta_9 : 6$ and $\zeta_{18} : 6$ and $[\tau_1, \dots, \tau_8] = [(\zeta_4 : 2).4]$, $[(\zeta_4 : 2).2, \zeta_{12} : 4]$ or $[(\zeta_{12} : 4).2]$. Then

$$[g^{6*}] = [1, (\zeta_3 : 2).3, \eta_1^6, \dots, \eta_7^6, -1.8]$$

where $[\eta_1^6, \dots, \eta_7^6] = [1.7]$ or $[1, (\zeta_3 : 2).3]$. Thus, $\text{Tr}(g^{6*} | H_{\text{et}}^2(X, \mathbb{Q}_l)) = -3$ or -12 . On the other hand, $\text{Fix}(g^6)$ is contained in the finite set $\text{Fix}(g^{18})$, so has a non-negative Euler number, i.e. $\text{Tr}(g^{6*} | H_{\text{et}}^2(X, \mathbb{Q}_l)) \geq -2$.

Claim: $\text{ord}(g) \neq 4.9$.

Suppose that $\text{ord}(g) = 4.9$. Then $[g^{9*}] = [1, 1.6, 1, (\zeta_4 : 2).4, -1.6]$, thus

$$[g^*] = [1, \zeta_9 : 6, 1, \eta_1, \dots, \eta_8, \tau_1, \dots, \tau_6]$$

where $[\eta_1, \dots, \eta_8] = [(\zeta_4 : 2).4]$, $[(\zeta_4 : 2).2, \zeta_{12} : 4]$ or $[(\zeta_{12} : 4).2]$, and $[\tau_1, \dots, \tau_6]$ consists of $-1, \zeta_6 : 2, \zeta_{18} : 6$. Since $\text{Fix}(g^3)$ being contained in the finite set $\text{Fix}(g^9)$ has a non-negative Euler number, $\text{Tr}(g^{3*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) \geq -2$. This determines $[\tau_1, \dots, \tau_6]$ uniquely, $[\tau_1, \dots, \tau_6] = [\zeta_{18} : 6]$. Then

$$[g^{6*}] = [1, (\zeta_3 : 2).3, 1, -1.8, (\zeta_3 : 2).3]$$

and $\text{Tr}(g^{6*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = -12$. But $\text{Fix}(g^6)$ being contained in the finite set $\text{Fix}(g^{18})$ has a non-negative Euler number.

Claim: $\text{ord}(g) \neq 4.7$.

Suppose that $\text{ord}(g) = 4.7$. Then $[g^{7*}] = [1, 1.6, 1, (\zeta_4 : 2).4, -1.6]$, thus

$$[g^*] = [1, \zeta_7 : 6, 1, (\zeta_4 : 2).4, \tau_1, \dots, \tau_6]$$

where $[\tau_1, \dots, \tau_6] = [\zeta_{14} : 6]$ or $[-1.6]$.

In the second case, $\text{Tr}(g^*|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 - 1 + 1 - 6 < -2$, contradicting $\text{Fix}(g) \subset \text{Fix}(g^7)$ a finite set.

In the first case, $[g^{2*}] = [1, \zeta_7 : 6, 1, -1.8, \zeta_7 : 6]$ and $\text{Tr}(g^{2*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = -8$. But $\text{Fix}(g^2)$ is contained in the finite set $\text{Fix}(g^{14})$.

Claim: $\text{ord}(g) \neq 2.14$.

Suppose that $\text{ord}(g) = 2.14$. Then $[g^{14*}] = [1, 1.6, 1.7, -1.8]$, thus

$$[g^*] = [1, \zeta_{14} : 6, \eta_1, \dots, \eta_7, (\zeta_4 : 2).4]$$

where $[\eta_1, \dots, \eta_7] = [\zeta_{14} : 6, \pm 1]$ or $[\zeta_7 : 6, \pm 1]$ or $[\pm 1, \dots, \pm 1]$. Since $\text{Fix}(g^2)$ being contained in the finite set $\text{Fix}(g^{14})$ has a non-negative Euler number, the first two possibilities for $[\eta_1, \dots, \eta_7]$ are removed. Then

$$\text{Tr}(g^{2*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 - 1 + 7 - 8 = -1$$

and $\text{Fix}(g^2)$ consists of a point. Thus the action of g on the 8-point set $\text{Fix}(g^{14})$ fixes one point and rotates the remaining 7 points. Then g^7 fixes the 8 points of $\text{Fix}(g^{14})$ and

$$\text{Tr}(g^{7*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = e(\text{Fix}(g^7)) - 2 = 6.$$

On the other hand, $[g^{7*}] = [1, -1.6, \eta_1, \dots, \eta_7, (\zeta_4 : 2).4]$, thus

$$\text{Tr}(g^{7*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = -5 + \sum_j \eta_j \leq 2.$$

□

Lemma 4.7. *Assume that $\phi(n) = 4$.*

- (1) *If $n = 12$, then $m \leq 3$ or $m = 5$.*
- (2) *If $n = 10$, then $m \leq 3$.*
- (3) *If $n = 8$, then $m \leq 3$ or $m = 5$.*
- (4) *If $n = 5$, then $m \leq 4$.*

Proof. We see that g^n is symplectic of order m with $[1, 1.4] \subset [g^{n*}]$, thus $m \leq 6$ by Lemma 2.6.

Assume that $m = 5$. Then $[g^{n*}] = [1, 1.5, (\zeta_5 : 4).4]$. If $n = 5$ or 10 , then ζ_{25} or $\zeta_{50} \in [g^*]$, but $\phi(25) = \phi(50) = 20 > 16$.

Assume that $\text{ord}(g) = 6.12$. Then $\text{ord}(g^2) = 6.6$, but such an order does not occur by Lemma 4.8.

Assume that $\text{ord}(g) = 4.12$. Then $\text{ord}(g^3) = 4.4$, but such an order does not occur by Lemma 4.8.

Assume that $\text{ord}(g) = 6.5$. Then $[g^{5*}] = [1, 1.4, 1, (\zeta_3 : 2).4, (\zeta_6 : 2).2, -1.4]$, thus

$$[g^*] = [1, \zeta_5 : 4, 1, \eta_1, \dots, \eta_8, (\zeta_6 : 2).2, \tau_1, \dots, \tau_4]$$

where $[\eta_1, \dots, \eta_8] = [(\zeta_3 : 2).4]$ or $[\zeta_{15} : 8]$, and $[\tau_1, \dots, \tau_4] = [\zeta_{10} : 4]$ or $[-1.4]$. Since $\text{Fix}(g)$ is contained in the finite set $\text{Fix}(g^5)$, we see that $-2 \leq \text{Tr}(g^*|H_{\text{et}}^2(X, \mathbb{Q}_l)) \leq \text{Tr}(g^{5*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 0$. Thus

$$[g^*] = [1, \zeta_5 : 4, 1, (\zeta_3 : 2).4, (\zeta_6 : 2).2, \zeta_{10} : 4]$$

or

$$[g^*] = [1, \zeta_5 : 4, 1, \zeta_{15} : 8, (\zeta_6 : 2).2, -1.4].$$

In the first case, $\text{Tr}(g^{2*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = -6$ contradicting $\text{Fix}(g^2) \subset \text{Fix}(g^{10})$ a finite set.

In the second, $\text{Tr}(g^{3*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = -9$ contradicting $\text{Fix}(g^3) \subset \text{Fix}(g^{15})$.

Assume that $\text{ord}(g) = 4.10$. Then $[g^{10*}] = [1, 1.7, (\zeta_4 : 2).4, -1.6]$, so

$$[g^*] = [1, \zeta_{10} : 4, \pm 1, \pm 1, \pm 1, (\zeta_8 : 4).2, (\zeta_4 : 2).3].$$

Thus $\text{Tr}(g^{2*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = -3$ contradicting $\text{Fix}(g^2) \subset \text{Fix}(g^{10})$ a finite set.

Assume that $\text{ord}(g) = 6.8$ or 4.8 . Then $\text{ord}(g^2) = 6.4$ or 4.4 , but such orders do not occur by Lemma 4.8. \square

Lemma 4.8. *Assume that $\phi(n) \leq 2$.*

- (1) *If $n = 6$, then $m \neq 6, 8$.*
- (2) *If $n = 4$, then $m \neq 4, 6, 8$.*
- (3) *If $n = 3$, then $m \neq 6$.*
- (4) *If $n = 2$, then $m \neq 8$.*

Proof. Assume that $\phi(n) = 2$ and $m = 6$. Then

$$[g^{n*}] = [1, 1.4, 1, (\zeta_3 : 2).4, (\zeta_6 : 2).2, -1.4].$$

We infer that $\zeta_6 \in [g^{n*}]$ must come from ζ_{6n} in $[g^*]$. But $\phi(6n) = 12, 8, 6$ respectively if $n = 6, 4, 3$. In any case, $\phi(6n) > 4$.

Assume that $\text{ord}(g) = 8.2$. Then $[g^{2*}] = [1, 1.3, (\zeta_8 : 4).2, (\zeta_4 : 2).3, -1.4]$. We infer that $(\zeta_4 : 2).3$ must come from $\zeta_8 : 4$ in $[g^*]$, a contradiction. This also shows that 8.4 and 8.6 do not occur.

Assume that $\text{ord}(g) = 4.4$. Then $[g^{4*}] = [1, 1.4, 1.3, (\zeta_4 : 2).4, -1.6]$. We infer that -1.6 must come from $\zeta_8 : 4$ in $[g^*]$, a contradiction. \square

5. THE COMPLEX CASE

Throughout this section, X is a complex K3 surface.

Assume that X is not projective. Then all automorphisms of finite order are symplectic, hence of order ≤ 8 , (see Section 3.)

Thus, we may assume that X is projective. Then the image of $\text{Aut}(X)$ on $\text{GL}(H^0(X, \Omega_X^2)) \cong \mathbb{C}^*$ is a finite cyclic group [20], and every automorphism of finite order has an invariant ample divisor class, hence its induced action on the 2nd integral cohomology $H^2(X, \mathbb{Z})$ has 1 as an eigenvalue. The proof of the only-if-part of Theorem 1.2 is just a copy of the tame case, once we replace $H_{\text{et}}^2(X, \mathbb{Q}_l)$ by $H^2(X, \mathbb{Z})$. Here, we also replace Proposition 2.4 by the usual Lefschetz fixed point formula from topology. Proposition 3.4 gives the if-part of Theorem 1.2.

We remark that in the complex case the holomorphic Lefschetz formula improves Lemma 2.7 to a finer form: X^h is empty. But we do not need this.

6. FAITHFULNESS OF THE REPRESENTATION ON THE COHOMOLOGY

In this section we prove Theorem 1.4.

Proof. Let g be an automorphism of X such that g^* acts on $H_{\text{et}}^2(X, \mathbb{Q}_l)$ trivially. By Proposition 2.1, g^* acts trivially on $H^0(X, \Omega_X^2)$, hence g is symplectic. From the exact sequence of \mathbb{Q}_l -vector spaces

$$(6.1) \quad 0 \rightarrow \text{NS}(X) \otimes \mathbb{Q}_l \rightarrow H_{\text{et}}^2(X, \mathbb{Q}_l) \rightarrow T_l^2(X) \rightarrow 0,$$

we see that g^* acts trivially on the Néron-Severi group $\text{NS}(X)$. As was pointed out by Rudakov and Shafarevich ([24], Sec. 8, Prop. 3), we may assume that g is of finite order. Indeed, automorphisms acting trivially on $\text{Pic}(X)$ form an algebraic group, and there is no non-zero regular vector field on a K3 surface ([24], Sec. 6, Theorem).

We will show that $g = 1_X$ the identity automorphism.

Case 1: $\text{ord}(g)$ is coprime to the characteristic p , i.e. g is tame. By Proposition 2.5, the representation of a tame symplectic automorphism on $H_{\text{et}}^2(X, \mathbb{Q}_l)$ is faithful. Hence our g must be the identity automorphism.

Case 2: $\text{ord}(g) = p^r m$ for some m coprime to p . Since g acts trivially on $H_{\text{et}}^2(X, \mathbb{Q}_l)$, so does g^{p^r} , which has order m . Hence by Case 1, $g^{p^r} = 1$.

Case 3: $\text{ord}(g) = p^r$. Since g acts trivially on $H_{\text{et}}^2(X, \mathbb{Q}_l)$, so is $g^{p^{r-1}}$. Thus we may assume that

$$\text{ord}(g) = p.$$

By Theorem 2.1 [7], an automorphism of order the characteristic p exists only if $p \leq 11$. By the result of [6] we know that the quotient surface $X/\langle g \rangle$ is a rational surface or an Enriques surface (non-classical of μ_2 -type) or a K3 surface with rational double points.

Assume that $X/\langle g \rangle$ is a rational surface. This case occurs when the fixed locus X^g is either a point giving a Gorenstein elliptic singularity on the

quotient surface or of 1-dimensional. In the latter case the quotient surface has rational singularities. Since $X/\langle g \rangle$ is rational, by Proposition 2.2

$$\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^g = \text{rank NS}(X)^g.$$

Since g^* acts trivially on $H_{\text{et}}^2(X, \mathbb{Q}_l)$, it acts trivially on $\text{NS}(X)$. Thus

$$\text{rank NS}(X) = \text{rank NS}(X)^g = \dim H_{\text{et}}^2(X, \mathbb{Q}_l)^g = 22.$$

Thus X is supersingular. For supersingular K3 surfaces, Ogus [23] proved the faithfulness of the representation of the automorphism group of a supersingular K3 surface X on the Néron-Severi group $\text{NS}(X)$.

Assume that $X/\langle g \rangle$ is an Enriques surface. This case occurs when $p = 2$ and X^g is empty. By Proposition 2.2, we have

$$\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^g = \text{rank NS}(X)^g.$$

Now just repeat the proof of the previous case.

Assume that $X/\langle g \rangle$ is a K3 surface with rational double points. This case occurs when the fixed locus X^g consists of either two points or a point which gives a rational double point on the quotient. Let $Y \rightarrow X/\langle g \rangle$ be a minimal resolution. Then Y is a K3 surface and

$$\text{rank NS}(Y) > \text{rank NS}(X/\langle g \rangle) = \text{rank NS}(X)^g.$$

On the other hand, by Proposition 5 of [26]

$$T_l^2(Y) \cong T_l^2(X)^g.$$

Combining these two, we get

$$\dim H_{\text{et}}^2(Y, \mathbb{Q}_l) = \text{rank NS}(Y) + \dim T_l^2(Y) > \text{rank NS}(X)^g + \dim T_l^2(X)^g.$$

But the right hand side is 22, since $g^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$ is trivial. \square

7. THE CASE: $p = 11$

In this section we assume that $p = 11$ and determine the orders of wild automorphisms. We first recall some previous results from [6], [7], [8].

Proposition 7.1. [6], [7], [8] *Let u be an automorphism of order 11 of a K3 surface X over an algebraically closed field of characteristic $p = 11$. Then*

- (1) *the fixed locus X^u is a cuspidal curve or a point;*
- (2) *X admits an u -invariant fibration $X \rightarrow \mathbb{P}^1$ of curves of genus 1, with or without a section, and u acts on the base \mathbb{P}^1 with one fixed point;*
- (3) *the corresponding fibre F_0 is of type II and the fixed locus X^u is either equal to F_0 or the cusp of F_0 ;*
- (4) *the action of u^* on $H_{\text{et}}^2(X, \mathbb{Q}_l)$, $l \neq 11$, has eigenvalues*

$$[u^*] = [1, 1, (\zeta_{11} : 10).2]$$

where the first eigenvalue corresponds to a u -invariant ample class.

Proof. (1)–(3) are contained in [6] and [7].

(4) By Proposition 4.2 [8], $\text{Tr}(u^*|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 0$. Since an eigenvalue of u^* is an 11-th root of unity, the result follows. \square

We now state the main result of this section.

Theorem 7.2. *Let g be an automorphism of finite order of a K3 surface X over an algebraically closed field of characteristic $p = 11$. Assume that the order of g is divisible by 11. Then*

$$\text{ord}(g) = 11n, \text{ where } n = 1, 2, 3, 6.$$

All these orders are realized by an example (see Example 7.5).

The proof follows from the following two lemmas 7.3 and 7.4.

Lemma 7.3. *$n \neq 9$ and n is not a prime > 3 .*

Proof. Suppose that $n = 9$ or a prime ≥ 5 . Then g^{n*} on $H_{\text{et}}^2(X, \mathbb{Q}_l)$, $l \neq 11$, has eigenvalues $[g^{n*}] = [1, 1, (\zeta_{11} : 10).2]$ where the first eigenvalue corresponds to an invariant ample divisor. Since $\phi(11n) > 20$, the eigenvalues $\zeta_{11n} : \phi(11n)$ cannot appear in $[g^*]$. Then by Theorem 1.4, the eigenvalues $\zeta_n : \phi(n)$ must appear in $[g^*]$. But $\phi(n) > 1$. \square

Lemma 7.4. *$n \neq 4$.*

Proof. Suppose that $n = 4$. By Theorem 1.4, g^* acting on $H_{\text{et}}^2(X, \mathbb{Q}_l)$ has ζ_{44} as an eigenvalue and hence

$$[g^*] = [1, \pm 1, \zeta_{44} : 20]$$

where the first eigenvalue corresponds to a g -invariant ample divisor. Let

$$s := g^{11}, \quad u := g^4.$$

Then s^2 is an involution of X with $[s^{2*}] = [1, 1, -1.20]$. Thus by Lemma 2.8, s^2 is non-symplectic and $X/\langle s^2 \rangle \cong \mathbb{F}_e$ a rational ruled surface. Applying Deligne-Lusztig (Proposition 2.3) to $g^{26} = s^2u$, we see that

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^*|H_{\text{et}}^j(X^{s^2}, \mathbb{Q}_l)) = \sum_{j=0}^4 (-1)^j \text{Tr}(g^{26*}|H_{\text{et}}^j(X, \mathbb{Q}_l)) = 6.$$

Assume that $\text{Fix}(s^2)$ is a curve C_9 of genus 9. The characteristic polynomial of $u^*|H_{\text{et}}^1(C_9, \mathbb{Q}_l)$ has integer coefficients and $(u^{11})^* = 1$, hence $u^*|H_{\text{et}}^1(C_9, \mathbb{Q}_l)$ has trace 18 or 7, since it is an integer. So

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^*|H_{\text{et}}^j(X^{s^2}, \mathbb{Q}_l)) = -16 \quad \text{or} \quad -5,$$

neither compatible with the above computation. Thus

$$\text{Fix}(s^2) = C_0 \cup C_{10}$$

where C_i is a curve of genus i . In this case Deligne-Lusztig does not work, as $u^*|H_{\text{et}}^1(C_{10}, \mathbb{Q}_l)$ may have trace -2 . We need a new argument. Note that the induced automorphism \bar{g} leaves invariant the unique ruling of \mathbb{F}_4 . Let

$$\psi : X \rightarrow \mathbb{P}^1$$

be the fibration of curves of genus 1 induced from the ruling on the quotient. It follows that g acts on the base \mathbb{P}^1 , and C_0 is a section of ψ and C_{10} a 3-section. Clearly $g^{44} = 1$ on \mathbb{P}^1 . The order 11 automorphism $u = g^4$ of X acts nontrivially on the base \mathbb{P}^1 ; otherwise it would be induced by the translation by an 11-torsion, but there is no p -torsion on an elliptic K3 surface in characteristic $p > 7$ (Theorem 2.13 [7]). By Lemma 2.11, $s = g^{11}$ acts trivially on the base \mathbb{P}^1 of ψ . Thus we infer that s acts trivially on both C_0 and C_{10} , i.e.,

$$X^s = C_0 \cup C_{10}.$$

Then $e(X^s) = -16$. On the other hand, from the list $[g^*]$ we compute that $[s^*] = [g^{11*}] = [1, 1, (\zeta_4 : 2).10]$, hence $\sum_{j=0}^4 (-1)^j \text{Tr}(s^*|H_{\text{et}}^j(X, \mathbb{Q}_l)) = 4$, contradicting Lefschets. \square

Example 7.5. In char $p = 11$, there are K3 surfaces with an automorphism of order 22

$$X_\varepsilon : y^2 + x^3 + \varepsilon x^2 + t^{11} - t = 0, \quad g_{\varepsilon, 22}(t, x, y) = (t + 1, x, -y).$$

When $\varepsilon = 0$, X_0 also admits an automorphism of order 66

$$g_{66} : (t, x, y) \mapsto (t + 1, \zeta_3 x, -y).$$

The surface X_ε has a II -fibre at $t = \infty$ and 22 I_1 -fibres at $(t^{11} - t)(t^{11} - t + 3\varepsilon^3) = 0$ if $\varepsilon \neq 0$, and 11 II -fibres at $t^{11} - t = 0$ if $\varepsilon = 0$ ([8], [6], 5.8).

In characteristic 11, it is known (Lemma 2.3 [8]) that a K3 surface admitting an automorphism of order 11 has Picard number 2, 12 or 22. For K3 surfaces with an automorphism of order 33, the second cannot occur.

Proposition 7.6. *In characteristic $p = 11$, if X admits an automorphism of order 33, then the Picard number $\rho(X) = 2$ or 22.*

Proof. Let g be an automorphism of order 33. Our method shows that the action of g^* on $H_{\text{et}}^2(X, \mathbb{Q}_l)$, $l \neq 11$, has eigenvalues $[g^*] = [1, 1, \zeta_{33} : 20]$. On the other hand, by Proposition 2.2 both traces of g^* on $\text{NS}(X)$ and on $T_l^2(X)$ are integers. Since $X/\langle g \rangle$ is a rational surface, $\dim T_l^2(X)^g = 0$. \square

8. THE CASE: $p = 7$

In this section we determine the orders of wild automorphisms in characteristic $p = 7$. We first improve the previous results from [6] and [7].

Proposition 8.1. *Let u be an automorphism of order 7 of a K3 surface X over an algebraically closed field of characteristic $p = 7$. Then*

- (1) *the fixed locus X^u is a connected curve or a point and $X/\langle u \rangle$ is a rational surface;*

- (2) X admits a u -invariant fibration $\psi : X \rightarrow \mathbb{P}^1$ of curves of arithmetic genus 1, with or without a section, and there is a specific fibre F_0 such that X^u is either the support of F_0 or a point of F_0 ;
- (3) if $X^u = \text{supp}(F_0)$, then u acts on \mathbb{P}^1 non-trivially and F_0 is of type II^* or III ;
- (4) if X^u is a point, then F_0 is of type III , X^u is the singular point of F_0 and
 - (a) u acts on \mathbb{P}^1 non-trivially; or
 - (b) u acts on \mathbb{P}^1 trivially, ψ has 3 singular fibres of type I_7 away from F_0 and u is induced by the translation by a 7-torsion of the Jacobian fibration of ψ ;
- (5) $u^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$, $l \neq 7$, has eigenvalues

$$[u^*] = [1, 1.9, (\zeta_7 : 6).2] \text{ or } [1, 1.3, (\zeta_7 : 6).3]$$

respectively if F_0 is of type II^* or III , where the first eigenvalue corresponds to a u -invariant ample divisor.

Proof. The assertions (1)–(3) are explicitly stated in [6] and [7]. We will prove the last two assertions.

(4) Assume that X^u is a point. Let F_0 be the fibre of ψ containing X^u .

Assume that u acts on \mathbb{P}^1 non-trivially. The singular fibres other than F_0 form orbits under the action of u , hence the sum of their Euler numbers is divisible by the characteristic. If the sum ≤ 14 , then F_0 has Euler number $e(F_0) \geq 10$. But no singular fibre with Euler number ≥ 10 admits an order 7 automorphism with just one fixed point. Thus the sum is 21. It follows that $e(F_0) = 3$ and hence F_0 is of type III or I_3 . Since a fibre of type I_3 does not admit an order 7 automorphism with one fixed point, F_0 is of type III and X^u is the singular point of F_0 .

Assume that u acts on \mathbb{P}^1 trivially. Then u is induced by the translation by a 7-torsion of the Jacobian fibration of ψ . Each singular fibre other than F_0 is acted on by u without fixed points, so must be of type I_{7m} . Let

$$I_{7m_1}, \dots, I_{7m_r}$$

be the types of all singular fibres other than F_0 . Then

$$e(F_0) = 24 - 7 \sum m_i.$$

Since F_0 admits an order 7 automorphism with one fixed point, this is possible only if $\sum m_i = 3$ and F_0 is of type III . To determine m_1, \dots, m_r , we look at the Jacobian fibration. It is known that the Jacobian fibration is again a K3 surface with singular fibres of the same type as ψ (see [2]). So, to determine the types of singular fibres, we may assume that ψ has a section and u is the translation by a 7-torsion of the fibration ψ . Applying the explicit formula for the height pairing ([3] or Theorem 8.6 [27]) on the Mordell-Weil group of an elliptic surface, or in our K3 case the formula given in [6], p.121, we deduce that $r = 3$ and $m_1 = m_2 = m_3 = 1$.

(5) Assume that F_0 is of type II^* . Then the 9 components of F_0 are fixed by u . The rational elliptic surface $X/\langle u \rangle$ is singular along the fibre \bar{F}_0 coming from F_0 , but the sum of Euler numbers of other singular fibres is 2. We infer that the fibre Y_0 of the relative minimal model Y of the minimal resolution of $X/\langle u \rangle$ corresponding to \bar{F}_0 must be of type II^* or I_4^* . Since the number of components of Y_0 is the number of components of \bar{F}_0 , we see that

$$\text{rank NS}(X/\langle u \rangle) = 10.$$

Then by Proposition 2.2, $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^u = \text{rank NS}(X)^u = 10$.

Assume that F_0 is of type III . Then the order 7 automorphism u preserves each of the two components of F_0 . The rational elliptic surface $X/\langle u \rangle$ is singular along the fibre \bar{F}_0 coming from F_0 , but the sum of Euler numbers of other singular fibres is 3. We infer that the fibre Y_0 of the relative minimal model Y of the minimal resolution of $X/\langle u \rangle$ corresponding to \bar{F}_0 must be of type III^* or I_3^* . Since the number of components of Y_0 is 8 while the number of components of \bar{F}_0 is 2, we see that

$$\text{rank NS}(X/\langle u \rangle) = 10 - 6 = 4.$$

Then by Proposition 2.2, $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^u = \text{rank NS}(X)^u = 4$. □

The following is the main result of this section.

Theorem 8.2. *Let g be an automorphism of finite order of a K3 surface X over an algebraically closed field of characteristic $p = 7$. Assume that the order of g is divisible by 7. Then*

$$\text{ord}(g) = 7n, \text{ where } n = 1, 2, 3, 4, 6.$$

All these orders are realized by an example (see Example 8.6).

The proof follows from the following lemmas 8.3—8.5.

Lemma 8.3. *$n \neq 9$, $n \neq 8$ and $n \neq$ a prime > 3 .*

Proof. Consider the order 7 automorphism

$$u := g^n.$$

Suppose that $n = 9$ or 8 or a prime ≥ 5 .

Case: $[u^*] = [1, 1.3, (\zeta_7 : 6).3]$. By Theorem 1.4, g^* on $H_{\text{et}}^2(X, \mathbb{Q}_l)$ should have ζ_n or ζ_{7n} as an eigenvalue. If $\zeta_n \in [g^*]$, then the eigenvalues $1.\phi(n)$ appear in $[u^*]$, but $\phi(n) > 3$. Since $\phi(7n) > 18$, $\zeta_{7n} \notin [g^*]$.

Case: $[u^*] = [1, 1.9, (\zeta_7 : 6).2]$. In this case X^u is the support of a II^* fibre F_0 . Since g acts on X^u , it preserves every component of F_0 . The 9 components and a g -invariant ample class are linearly independent in the Néron-Severi group. Thus $\text{rank NS}(X)^g = 10$. Then by Theorem 1.4, $\zeta_{7n} \in [g^*]$, but $\phi(7n) > 12$. □

Lemma 8.4. *If $n = 4$, then*

$$[g^{4*}] = [1, 1.3, (\zeta_7 : 6).3].$$

Proof. Consider the order 7 automorphism

$$u := g^4.$$

Suppose that $[u^*] = [1, 1.9, (\zeta_7 : 6).2]$ and $\text{Fix}(u)$ is the support of a type II^* fibre F_0 of a fibration $\psi : X \rightarrow \mathbb{P}^1$. As in the proof of Lemma 8.3, we infer that $\text{rank NS}(X)^g = 10$. Thus by Theorem 1.4,

$$[g^*] = [1, 1.9, \zeta_{28} : 12].$$

Let

$$s := g^7.$$

Then s is a tame automorphism of order 4 with $[s^*] = [1, 1.9, (\zeta_4 : 2).6]$. By Proposition 2.4, $e(X^s) = 12$. Thus X^s is non-empty. The involution $s^2 = g^{14}$ has $[s^{2*}] = [g^{14*}] = [1, 1.9, -1.12]$, hence $e(\text{Fix}(s^2)) = 0$. Since $\text{Fix}(s^2)$ contains $\text{Fix}(s)$, it is non-empty. Let us determine $\text{Fix}(s^2)$. Let R_1, R_2, \dots, R_9 be the components of X^u .

$$\begin{array}{c} R_8 - R_7 - R_6 - R_5 - R_4 - R_3 - R_2 - R_1 \\ \quad \quad \quad \downarrow \\ \quad \quad \quad R_9 \end{array}$$

Note that s^2 acts on X^u as an involution. Any involution of \mathbb{P}^1 in $\text{char} \neq 2$ has exactly two fixed points. The component R_6 is acted on by s^2 with 3 fixed points, hence fixed point-wisely by s^2 . Note that a non-symplectic involution of a K3 surface has no isolated fixed points. If R_8 is not fixed point-wisely by s^2 , then a curve $C \subset \text{Fix}(s^2)$ must pass through the intersection point of R_8 and R_7 , then locally at the intersection of the 3 curves s^2 preserves R_8 and R_7 and fixes C point-wisely, which is impossible in any characteristic for a local tame automorphism of a 2-dimensional space. Thus R_8 is fixed point-wisely. Similarly, R_4 and R_2 are fixed point-wisely. Then there is a curve $C \subset \text{Fix}(s^2)$ intersecting R_9 and R_1 each with multiplicity 1. The curve C is either irreducible or a union of two curves respectively intersecting R_9 and R_1 . We claim that

$$\text{Fix}(s^2) = R_2 \cup R_4 \cup R_6 \cup R_8 \cup C.$$

Indeed, if there is another component of X^{s^2} , then it does not meet F_0 , hence must be contained in a fibre. But any fibre different from F_0 is irreducible (Proposition 8.1). Computing $[g^{18*}] = [1, 1.9, (\zeta_{14} : 6).2]$ and applying Deligne-Lusztig (Proposition 2.3) to $g^{18} = s^2u$, we get

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(X^{s^2}, \mathbb{Q}_l)) = \sum_{j=0}^4 (-1)^j \text{Tr}(g^{18*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) = 14.$$

Assume that C is irreducible. Then it is a curve C_5 of genus 5, since $e(\text{Fix}(s^2)) = 0$. It is easy to see that $\text{Tr}(u^*|H_{\text{et}}^1(C_5, \mathbb{Q}_l)) = 10$ or 3. Thus

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^*|H_{\text{et}}^j(X^{s^2}, \mathbb{Q}_l)) = 0 \quad \text{or} \quad 7,$$

neither compatible with the above computation. Thus

$$C = C_0 \cup C_6$$

where C_i is a curve of genus i . In this case Deligne-Lusztig does not work, as it may happen that $\text{Tr}(u^*|H_{\text{et}}^1(C_6, \mathbb{Q}_l)) = -2$. We employ a new argument. We infer that C_0 meets R_1 , hence is a section, and C_6 meets R_9 , hence is a 3-section of $\psi : X \rightarrow \mathbb{P}^1$. It follows that s^2 acts on a general fibre of ψ with 4 fixed points, the intersection of C and the fibre. Let

$$Y \rightarrow X/\langle s^2 \rangle$$

be the minimal resolution. Then Y is a rational ruled surface, not relatively minimal. Note that $g(X^u) = X^u$ and hence g preserves the fibration $\psi : X \rightarrow \mathbb{P}^1$. Clearly $g^{28} = 1$ on the base \mathbb{P}^1 . By Proposition 8.1, g^4 acts non-trivially on the base \mathbb{P}^1 . Thus $s = g^7$ acts trivially on the base \mathbb{P}^1 by Lemma 2.11. On the other hand, s acts on $\text{Fix}(s^2)$ and on X^u , hence on both C_0 and C_6 . Since C_0 is a section of ψ and C_6 a 3-section, we see that s acts trivially on both C_0 and C_6 , i.e.,

$$X^s \supset C_0 \cup C_6.$$

Thus s induces an involution \bar{s} on Y , which leaves invariant each fibre of Y . Moreover, on a general fibre of Y \bar{s} has 4 fixed points, but no involution of \mathbb{P}^1 can have 4 fixed points. \square

Lemma 8.5. $n \neq 12$.

Proof. Suppose that $n = 12$. Define

$$u := g^{12}.$$

Then u has order 7. Applying Lemma 8.4 to g^3 , we see that

$$[u^*] = [1, 1.3, (\zeta_7 : 6).3].$$

Since $\phi(84) > 18$, $\zeta_{84} \notin [g^*]$. By Theorem 1.4, we infer that the list of eigenvalues of $[g^*]$ is one of the following:

$$[g^*] = [1, \pm 1, \pm(\zeta_3 : 2), \zeta_{28} : 12, \pm(\zeta_7 : 6)],$$

$$[g^*] = [1, \pm 1, \zeta_4 : 2, \pm(\zeta_{21} : 12), \pm(\zeta_7 : 6)],$$

where $\pm(\zeta_{21} : 12)$ means $\zeta_{21} : 12$ or $\zeta_{42} : 12$. In any case, we claim that

$$\text{Fix}(u) = \text{Fix}(g^{12}) = \{\text{a point}\}.$$

Indeed, if $\text{Fix}(u)$ is the support of a fibre F_0 of type *III*, then the two components of F_0 are interchanged or preserved by g , so the eigenvalues $[1, 1, \pm 1]$ should appear in $[g^*]$, then $\zeta_3, \zeta_4, \zeta_6 \notin [g^*]$, hence $\zeta_{84} \in [g^*]$, impossible.

Assume the first case $[g^*] = [1, \pm 1, \pm(\zeta_3 : 2), \zeta_{28} : 12, \pm(\zeta_7 : 6)]$. The tame involution $s := g^{42}$ has

$$[s^*] = [g^{42*}] = [1, 1, 1.2, -1.12, 1.6],$$

hence it is non-symplectic and by Proposition 2.4 $e(\text{Fix}(g^{42})) = 0$. If g^{42} acts freely on X , then so does g^{21} . But no K3 surface admits an order 4 free action. Thus $\text{Fix}(g^{42})$ is either a union of elliptic curves or a union of a curve of genus $d+1 \geq 2$ and d smooth rational curves. Applying Deligne-Lusztig (Proposition 2.3) to $g^{54} = su$, we get

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(X^s, \mathbb{Q}_l)) = \sum_{j=0}^4 (-1)^j \text{Tr}(g^{54*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) = 7.$$

Any order 7 or trivial action on an elliptic curve E has

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(E, \mathbb{Q}_l)) = 1 - 2 + 1 = 0.$$

If 7 elliptic curves form an orbit under u , then the trace on the j -th cohomology of the union of the 7 curves is 0. This rules out the first possibility for $\text{Fix}(g^{42})$. Thus $\text{Fix}(g^{42})$ is a union of a curve C_{d+1} of genus $d+1 \geq 2$ and d smooth rational curves. Since fixed curves give linearly independent invariant vectors in $H_{\text{et}}^2(X, \mathbb{Q}_l)$ and $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 10$,

$$1 \leq d \leq 9.$$

Consider the action of $u = g^{12}$ on $\text{Fix}(g^{42})$. It is of order 7 or trivial. Since $\text{Fix}(g^{12})$ is a point, the action of u on $\text{Fix}(g^{42})$ has at most one fixed point and is of order 7. This is possible only if

- (1) $d = 7r$ and the d smooth rational curves form r orbits under u ; or
- (2) $d = 7r + 1$, u fixes one point of a smooth rational curve and the remaining $d - 1$ rational curves form r orbits under u .

In case (1), $d = 7$ and u acts on C_8 . Since $\text{Tr}(u^* | H_{\text{et}}^1(C_8, \mathbb{Q}_l)) = 16 - 7b$, $0 \leq b \leq 2$, we see that

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(X^s, \mathbb{Q}_l)) = 0 + 2 - (16 - 7b) = 7b - 14 \neq 7.$$

In case (2), $d = 1$ or 8 and u acts freely on C_2 or C_9 , but no genus 2 or 9 curve admits an order 7 free action in any characteristic.

Assume the second case $[g^*] = [1, \pm 1, \zeta_4 : 2, \pm(\zeta_{21} : 12), \pm(\zeta_7 : 6)]$. The tame involution $s := g^{42}$ has

$$[s^*] = [g^{42*}] = [1, 1, -1.2, 1.12, 1.6],$$

hence it is non-symplectic and by Proposition 2.4 $e(\text{Fix}(g^{42})) = 20$. Thus $\text{Fix}(g^{42})$ is either a union of 10 smooth rational curves and possibly some elliptic curves or a union of a curve of genus $d-9 \geq 2$ and d smooth rational curves. In the first case, the order 7 action of g^6 on $\text{Fix}(g^{42})$ preserves at

least 3 smooth rational curves, hence fixes at least 3 points, a contradiction. Thus we have the second case. Since fixed curves give linearly independent invariant vectors in $H_{\text{et}}^2(X, \mathbb{Q}_l)$ and $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 20$,

$$11 \leq d \leq 19.$$

Consider the action of g^6 on $\text{Fix}(g^{42})$. It is of order 7 or trivial. Since $\text{Fix}(g^6) \subset \text{Fix}(g^{12})$, the action of g^6 on $\text{Fix}(g^{42})$ has exactly one fixed point and is of order 7. This is possible only if

- (1) $d = 7r$ and the d smooth rational curves form r orbits under g^6 ; or
- (2) $d = 7r + 1$, u fixes one point of a smooth rational curve and the remaining $d - 1$ rational curves form r orbits under g^6 .

In any case, $r = 2$ and each orbit gives a g^6 -invariant divisor, hence

$$\text{rank NS}(X)^{g^6} \geq 3.$$

But $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^{g^6} = 2$. □

Example 8.6. In char $p = 7$, there are K3 surfaces with an automorphism of order 42 or 28.

- (1) $X_{42} : y^2 = x^3 + t^7 - t$, $g_{42}(t, x, y) = (t + 1, \zeta_3 x, -y)$;
- (2) $X_{28} : y^2 = x^3 + (t^7 - t)x$, $g_{28}(t, x, y) = (t + 1, -x, \zeta_4 y)$.

The surface X_{42} has a II^* -fibre at $t = \infty$ and 7 II -fibres at $t^7 - t = 0$; X_{28} has 8 III -fibres at $t = \infty$, $t^7 - t = 0$ ([6], 5.8).

9. THE CASE: $p = 5$

In this section we determine the orders of wild automorphisms in characteristic $p = 5$. We first improve the previous results from [6] and [7].

Proposition 9.1. *Let u be an automorphism of order 5 of a K3 surface X over an algebraically closed field of characteristic $p = 5$. Then one of the following occurs:*

- (1) *the fixed locus X^u contains a curve of arithmetic genus 2;*
- (2) *X^u is the support of a fibre F_0 of a fibration $\psi : X \rightarrow \mathbb{P}^1$ of curves of arithmetic genus 1, with or without a section, F_0 is of type IV or III^* , and $u^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$, $l \neq 5$, has eigenvalues*

$$[u^*] = [1, 1.5, (\zeta_5 : 4).4] \text{ or } [1, 1.9, (\zeta_5 : 4).3]$$

respectively if F_0 is of type IV or III^ ;*

- (3) *X^u consists of two points, $X/\langle u \rangle$ is a K3 surface with two rational double points of type E_8 , and $u^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$ has eigenvalues*

$$[u^*] = [1, 1.5, (\zeta_5 : 4).4];$$

- (4) *X^u consists of a point, $X/\langle u \rangle$ is a K3 surface with one rational double point of type E_8 , X does not admit a u -invariant elliptic fibration, and $u^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$ has eigenvalues*

$$[u^*] = [1, 1.13, (\zeta_5 : 4).2];$$

- (5) X^u consists of a point, $X/\langle u \rangle$ is a rational surface, X admits a u -invariant elliptic fibration with a fibre F_0 of type IV whose singular point is X^u , and $u^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$ has eigenvalues

$$[u^*] = [1, 1.5, (\zeta_5 : 4).4]$$

where the first eigenvalue corresponds to a u -invariant ample divisor.

Proof. By [6], the fixed locus X^u consists of a point, two points or a connected curve of Kodaira dimension $\kappa(X, X^u) = 0, 1, 2$. By Proposition 2.5 [7], the case with $\kappa(X, X^u) = 0$ occurs only in characteristic 2, so does not occur in characteristic 5.

Assume that X^u is a connected curve of Kodaira dimension $\kappa(X, X^u) = 2$. Then by Proposition 2.3 [7], X^u contains a curve of arithmetic genus 2. This gives the case (1).

Assume that X^u is a connected curve of Kodaira dimension $\kappa(X, X^u) = 1$. Then by Theorem 3 [6], X^u is the support of a fibre F_0 of a fibration

$$\psi : X \rightarrow \mathbb{P}^1$$

of curves of arithmetic genus 1, with or without a section, the induced action of u on the base \mathbb{P}^1 is of order 5 and the quotient $X/\langle u \rangle$ is a (singular) rational elliptic surface. Let

$$Y \rightarrow X/\langle u \rangle$$

be a minimal resolution. Then Y is a rational elliptic surface, not necessarily relatively minimal. Let \overline{Y} be the relatively minimal rational elliptic surface. Denote by $Y_0 \subset Y$ and $\overline{Y}_0 \subset \overline{Y}$ be the fibres coming from F_0 . Since the singular fibres of ψ away from F_0 form orbits under u , we have

$$e(F_0) = 24 - 5r$$

where r is the sum of the Euler numbers of singular fibres of Y away from Y_0 . If $r \leq 1$, then $e(Y_0) \geq e(Y) - 1$, but no rational elliptic surface may have a fibre with that big Euler number. Thus $r \geq 2$.

Case: $r = 4$. Then $e(F_0) = 24 - 5r = 4$ and $e(\overline{Y}_0) = e(\overline{Y}) - r = 8$. Since F_0 is of type I_4 or IV , we infer that

$$10 = \text{rank NS}(\overline{Y}) = 4 + \text{rank NS}(X/\langle u \rangle).$$

Then by Proposition 2.2, $\dim H_{\text{et}}^2(Y, \mathbb{Q}_l)^u = \text{rank NS}(X/\langle u \rangle) = 6$. Thus $[u^*] = [1, 1.5, (\zeta_5 : 4).4]$. A monodromy argument excludes the possibility I_4 ; if F_0 is of type I_n or I_m^* , i.e., has a stable reduction of multiplicative type, then one can use Tate's analytic uniformization to compute the representation of the Galois group of the generic point of the strict local scheme on the group of l^t -torsion points (the l -adic representation), which leads to the monodromy computation. A fibre of type I_{5n} or I_{5m}^* would be too big for a rational elliptic surface if $n > 1$ or $m > 0$.

Case: $r = 3$. Then $e(F_0) = 24 - 5r = 9$ and $e(\overline{Y_0}) = e(\overline{Y}) - r = 9$. Since F_0 is of type I_9 , I_3^* , or III^* , we infer that

$$10 = \text{rank NS}(\overline{Y}) = \text{rank NS}(X/\langle u \rangle).$$

Then by Proposition 2.2, $\dim H_{\text{et}}^2(Y, \mathbb{Q}_l)^u = \text{rank NS}(X/\langle u \rangle) = 10$. Thus $[u^*] = [1, 1.9, (\zeta_5 : 4).3]$. A monodromy argument excludes the possibilities I_9 and I_3^* .

Case: $r = 2$. Then $e(F_0) = 24 - 5r = 14$ and $e(\overline{Y_0}) = e(\overline{Y}) - r = 10$. Thus F_0 is of type I_{14} or I_8^* , both are excluded by a monodromy argument.

Assume that X^u consists of two points. Then by Theorem 1 [6], the quotient $X/\langle u \rangle$ has two rational double points and the minimal resolution

$$Y \rightarrow X/\langle u \rangle$$

is a K3 surface. From a result of Artin [1] we see that each singularity of $X/\langle u \rangle$ must be of type E_8 , thus

$$\text{rank NS}(Y) = 16 + \text{rank NS}(X/\langle u \rangle) = 16 + \text{rank NS}(X)^u.$$

Now by Proposition 2.2

$$22 = \dim H_{\text{et}}^2(Y, \mathbb{Q}_l) = \text{rank NS}(Y) + \dim T_l^2(X)^u = 16 + H_{\text{et}}^2(X, \mathbb{Q}_l)^u,$$

hence $H_{\text{et}}^2(X, \mathbb{Q}_l)^u = 6$. This gives the case (3).

Assume that X^u consists of a point. Then by Theorem 1 [6], the minimal resolution Y of $X/\langle u \rangle$ is either a K3 surface or a rational surface. Suppose that X admits a u -invariant fibration of curves of arithmetic genus 1. Let F_0 be the fibre containing the point X^u . Since the order 5 action of u on F_0 has only one fixed point, we see that F_0 is of type II , III or IV . On the other hand, $e(F_0) = 24 - 5r$ which holds true even when u acts on the base trivially, where r is the sum of the Euler numbers of singular fibres of Y away from the singular fibre coming from F_0 . Thus F_0 is of type IV and $r = 4$. If $X/\langle u \rangle$ is not rational, then the minimal resolution Y is a K3 surface and the fibre Y_0 consists of the proper image of the 3 components of F_0 and the 8 components lying over the singularity of type E_8 , hence $e(Y_0) < 20 = e(Y) - 4$, a contradiction. Thus $X/\langle u \rangle$ is rational, yielding the case (5). Suppose that X does not admit a u -invariant elliptic fibration. Then by Proposition 2.9 [7], Y cannot be a rational surface. Thus Y is a K3 surface and we have the case (4). \square

Remark 9.2. The case (3) is also supported by examples. In fact, there are 2 dimensional family of elliptic K3 surfaces with a 5-torsion and with 2 fibres of type II and 4 fibres type I_5 ([12] Theorem 4.4). The automorphism induced by a 5-torsion has 2 fixed points, the cusps of the two type II -fibres.

The following is the main result of this section.

Theorem 9.3. *Let g be an automorphism of finite order of a K3 surface X over an algebraically closed field of characteristic $p = 5$. Assume that the order of g is divisible by 5. Then*

$$\text{ord}(g) = 5n, \text{ where } n = 1, 2, 3, 4, 6, 8.$$

There are examples supporting these orders (see Examples 9.5 and 9.11).

The proof follows from the following lemmas 9.4, 9.6 — 9.10.

Lemma 9.4. *Let g be an automorphism of order $5n$. Let $u = g^n$. If the fixed locus X^u contains a curve of arithmetic genus 2, then $n \leq 8$, $n \neq 5, 7$.*

Proof. Assume that X^u contains a curve C of arithmetic genus 2. Then by [7] Proposition 2.3, the linear system $|C|$ defines a double cover $X \rightarrow \mathbb{P}^2$ and X is birationally isomorphic to the surface

$$(9.1) \quad z^2 = (y^5 - yx^4)P_1(x_0, x) + P_6(x_0, x)$$

where $(x_0 : x : y)$ is the homogeneous coordinates of \mathbb{P}^2 , $P_i(x_0, x)$ is a homogeneous polynomial of degree i , and the induced automorphism \bar{u} of \mathbb{P}^2 is given by

$$\bar{u}(x_0, x, y) = (x_0, x, x + y).$$

Since g preserves $|C|$, it induces a linear automorphism \bar{g} of \mathbb{P}^2 . It follows that n is not a multiple of 5, since \mathbb{P}^2 in characteristic p cannot admit an automorphism of order p^2 . Note that g is induced from \bar{g} , possibly composed with the covering involution. Since $\bar{g}^n = \bar{u}$, either \bar{g} has order $5n$ or $\frac{5n}{2}$. The latter happens if and only if n is even and $g^{\frac{5n}{2}}$ is the covering involution. Since g preserves the equation (9.1), \bar{g} preserves the right hand side of (9.1) up to a scalar multiple.

Suppose that \bar{g} has order $5n$. Using a Jordan canonical form we may assume that

$$\bar{g} = \begin{pmatrix} \zeta_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{n} & 1 \end{pmatrix}$$

where ζ_n is a primitive n -th root of 1. If $n \geq 7$, then it is easy to see that there are two possibilities for the pair (P_1, P_6)

$$(P_1, P_6) = (Ax, Bx^6) \text{ or } (Ax_0, Bx_0x^5),$$

and in either case the point $(x_0 : x : y, z) = (1 : 0 : 0, 0)$ is a singular point that is not a rational double point, hence the equation (9.1) does not define a K3 surface. Thus $n \leq 6$, $n \neq 5$. If $n = 6$, then there are two possibilities for the pair

$$(P_1, P_6) = (Ax, B_0x_0^6 + B_6x^6) \text{ or } (Ax_0, Bx_0x^5).$$

The second case does not define a K3 surface. Neither does the first case with $B_0 = 0$. In the first case with $B_0 \neq 0$, the equation (9.1) is nonsingular and g maps z to $\pm z$. Either case defines an automorphism of order 30.

Suppose that $n = 2m$ and $\bar{g}^{5m} = 1$. Then $\bar{g}^m = \bar{u}^3$ and we may assume that

$$\bar{g} = \begin{pmatrix} \zeta_m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{3}{m} & 1 \end{pmatrix}.$$

If $m \geq 7$, then a similar computation would imply that the equation (9.1) does not define a K3 surface. If $m = 6$, then, as we saw in the above, the order of g would be $5m$, a contradiction. Thus $m \leq 4$. \square

Example 9.5. As we saw in the above proof, the K3 surface

$$z^2 = Ax(y^5 - yx^4) + x_0^6 + Bx^6$$

admits an automorphism g of order 30

$$g(x_0 : x : y, z) = (\zeta_6 x_0 : x : \frac{x}{6} + y, \pm z).$$

If $m = 4$, then there are two possibilities for the pair

$$(P_1, P_6) = (Ax, B_2 x_0^4 x^2 + B_6 x^6) \text{ or } (Ax_0, B_1 x_0^5 x + B_5 x_0 x^5).$$

Assume the second case with $B_1 \neq 0$. Then the equation (9.1) has 5 rational double points of type A_1 , hence defines a K3 surface. In this case g must map z^2 to $\zeta_4 z^2$, hence by putting $\zeta_4 = \zeta_8^2$ we can rewrite g as

$$g(x_0 : x : y, z) = (\zeta_8^2 x_0 : x : \frac{x}{4} + y, \zeta_8 z)$$

which is of order 40.

Lemma 9.6. *There is no automorphism of order 5^2 .*

Proof. Suppose that $\text{ord}(g) = 25$. Then $g^*|H_{\text{et}}^2(X, \mathbb{Q}_l), l \neq 5$, has eigenvalues

$$[g^*] = [1, 1, \zeta_{25} : 20].$$

Thus $u = g^5$ has $[u^*] = [1, 1, (\zeta_5 : 4).5]$, which does not occur by Proposition 9.1 and Lemma 9.4. \square

Lemma 9.7. *Let g be an automorphism of order $5n$, $5 \nmid n$. If $\kappa(X, X^u) = 1$ for $u = g^n$, then $n = 1, 2, 3, 4, 6$.*

Proof. Since $\kappa(X, X^u) = 1$, we are in the case (2) of Proposition 9.1. The fixed locus X^u is the support of a fibre F_0 of a fibration of curves of arithmetic genus 1, and F_0 is of type *IV* or *III**. Note that g acts on X^u .

Case 1. F_0 is of type *IV*. Then $u = g^n$ has

$$[u^*] = [1, 1.5, (\zeta_5 : 4).4].$$

Claim: $n \neq 3^2$, n cannot be a prime ≥ 7 .

Suppose that $n = 3^2$. Then by Theorem 1.4, either ζ_9 or ζ_{45} must appear in $[g^*]$. But $\phi(9) > 5$ and $\phi(45) > 16$. Similarly, n cannot be a prime ≥ 7 .

Claim: $n \neq 2^3$.

Suppose that $n = 2^3$. Then g preserves one component of X^u and interchanges or preserves the other two components. Thus the action of g on the 3 components of X^u has eigenvalues $[1, 1, \pm 1]$ and we infer that

$$[g^*] = [1, 1, 1, \pm 1, \eta_1, \eta_2, \zeta_{40} : 16]$$

where η_1, η_2 is a combination of $\zeta_4 : 2, -1, 1$. The involution $s = g^{20}$ preserves each of the three components of F_0 . It is tame and has $e(X^s) \neq 8$, hence is non-symplectic. Thus there is a curve C in $\text{Fix}(s)$ passing through the singular point of F_0 . The curve C may be equal to one of the 3 components of F_0 . In any case we get a contradiction since a tame involution, locally at a point of C , can preserve at most one curve other than C .

Claim: $n \neq 12$.

Suppose that $n = 12$. Then the tame involution $s = g^{30}$ preserves each of the three components of F_0 . Assume that $s = g^{30}$ is non-symplectic. Then we get a contradiction exactly the same as in the previous case. Assume that $s = g^{30}$ is symplectic. Then s has 8 fixed points on X by Proposition 2.5. We infer that it has 4 fixed points on F_0 . Since $g(F_0) = F_0$, g acts on the base of the elliptic fibration. Since $u = g^8$ acts non-trivially on the base, by Lemma 2.11 g^5 acts trivially. Thus $s = g^{30}$ acts on each singular fibre. There are at least 5 singular fibres away from F_0 and on each of them s has a fixed point.

Case 2. F_0 is of type III^* . Then $u = g^n$ has

$$[u^*] = [1, 1.9, (\zeta_5 : 4).3].$$

Claim: $n \neq 3^2$, $n \neq$ a prime ≥ 7 .

Suppose that $n = 3^2$. The order 9 action of g on X^u preserves each of the 8 components. Thus $[1, 1.8] \subset [g^*]$, hence $\zeta_9 \notin [g^*]$ and by Theorem 1.4 $\zeta_{45} \in [g^*]$. But $\phi(45) > 12$. Similarly, n cannot be a prime ≥ 7 .

Claim: $n \neq 2^3$.

Suppose that $n = 2^3$. Then g acts on X^u , flipping or preserving the configuration of the 8 components. Thus the action of g on the 8 components of X^u has eigenvalues $[1.8]$ or $[1.5, -1.3]$, hence $\zeta_8 \notin [g^*]$. Then by Theorem 1.4, $\zeta_{40} \in [g^*]$. But $\phi(40) > 12$.

Claim: $n \neq 12$.

Suppose that $n = 12$. As in the previous case, the action of g on the 8 components of X^u has eigenvalues $[1.8]$ or $[1.5, -1.3]$, hence $[g^*]$ contains either $[1, 1.8, \pm 1]$ or $[1, 1.5, -1.3, \pm 1]$. Since $\phi(60) > 12$, we infer that $[(\zeta_5 : 4).3]$ in $[u^*]$ must come from a combination of $\zeta_{30} : 8, \zeta_{20} : 8, \zeta_{15} : 8, \zeta_{10} : 4, \zeta_5 : 4$ in $[g^*]$. In any case, $g^{30} = 1$ or $g^{20} = 1$. \square

Lemma 9.8. *Let g be an automorphism of order $5n$, $5 \nmid n$. If g^n fixes exactly two points, then $n = 1, 2, 3, 4, 6$.*

Proof. Let $u := g^n$. Then u has order 5 and by Proposition 9.1

$$[u^*] = [1, 1.5, (\zeta_5 : 4).4]$$

where the first eigenvalue corresponds to a g -invariant ample divisor.

Claim: $n \neq 3^2$, n cannot be a prime ≥ 7 .

The proof is the same as in Case 1, Lemma 9.7.

Claim: $n \neq 2^3$.

Suppose that $n = 2^3$. Then $u = g^8$. Since g acts on the two point set $\text{Fix}(u)$ and $\text{Fix}(g^2) \subset \text{Fix}(g^4) \subset \text{Fix}(g^8)$, we see that

$$\text{Fix}(g^2) = \text{Fix}(g^4) = \text{Fix}(g^8) = \{\text{two points}\}.$$

The proof will be divided into 3 cases according to the possibility of $[g^*]$.

Case 1. $[g^*] = [1, \zeta_8 : 4, \pm 1, \zeta_{40} : 16]$.

The tame involution $s = g^{20}$ has $e(X^s) = -16$ and $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 2$, hence is non-symplectic. Thus X^s consists of either a curve C_9 of genus 9 or a smooth rational curve R and a curve C_{10} of genus 10. The tame automorphism g^{10} has $e(\text{Fix}(g^{10})) = 4$. Since $\text{Fix}(g^{10}) \subset X^s$, we infer that $\text{Fix}(g^{10})$ consists of either 4 points or the curve R and 2 points on C_{10} . In any case, g acts on $\text{Fix}(g^{10})$, hence g^2 fixes at least 3 points, a contradiction.

Case 2. $[g^*] = [1, \eta_1, \dots, \eta_5, \zeta_{40} : 16]$ where η_1, \dots, η_5 is a combination of $\zeta_4 : 2, -1, 1$. In this case, $s = g^{20}$ has $e(X^s) = -8$ and $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 6$, hence is non-symplectic. Thus X^s consists of a curve C_{d+5} of genus $d+5$ and d smooth rational curves, where $0 \leq d \leq 5$. Applying Deligne-Lusztig (Proposition 2.3) to $g^{28} = su$, we get

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(X^s, \mathbb{Q}_l)) = \sum_{j=0}^4 (-1)^j \text{Tr}(g^{28*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) = 12.$$

If $d \leq 2$, then $\sum_j (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(C_{d+5}, \mathbb{Q}_l)) \leq 5$. Thus we may assume that $d \geq 3$. If $d = 3$ or 4, then g^4 preserves each of the d smooth rational curves, hence fixes at least d points, a contradiction. If $d = 5$, then $u = g^8$ preserves each of the 5 smooth rational curves, since

$$\sum (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(C_{10}, \mathbb{Q}_l)) \leq 7.$$

Thus the 5 smooth rational curves does not form a single orbit under g . Then g^4 preserves each of them, hence fixes at least 5 points, a contradiction.

Case 3. $[g^*] = [1, \zeta_8 : 4, \pm 1, \eta_1, \dots, \eta_{16}]$ where η_1, \dots, η_{16} is a combination of $\zeta_{20} : 8, \zeta_{10} : 4, \zeta_5 : 4$. In this case, $s = g^{20}$ has $e(X^s) = 16$ and $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 18$, hence is non-symplectic. Thus X^s is either a union of 8 smooth rational curves and possibly some elliptic curves or a union of a curve C_{d-7} of genus $d-7$ and d smooth rational curves, where $9 \leq d \leq 17$. In the first case, the order 5 action of $u = g^8$ on $\text{Fix}(g^{20})$ preserves at least 3 among the 8 smooth rational curves, hence fixes at least 3 points, a contradiction. Thus we have the second case. Deligne-Lusztig (Proposition 2.3)

does not work here, so we need a different argument. The action of g^4 on X^s is of order 5 with exactly 2 fixed points. This is possible only if

- (1) $d = 5r$ and the d smooth rational curves form r orbits under g^4 ; or
- (2) $d = 5r + 1$, g^4 fixes one point of a smooth rational curve and the remaining $d - 1$ rational curves form r orbits under g^4 ; or
- (3) $d = 5r + 2$, g^4 fixes one point each of two smooth rational curves and the remaining $d - 2$ rational curves form r orbits under g^4 .

In any case, $r = 2$ or 3 and each orbit gives a g^4 -invariant divisor, hence $\text{rank NS}(X)^{g^4} \geq 3$. But $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^{g^4} = 2$.

Claim: $n \neq 12$.

Suppose that $n = 12$. Then $u = g^{12}$. Since g acts on the two point set $\text{Fix}(u)$ and $\text{Fix}(g^i) \subset \text{Fix}(g^{12})$ for any i dividing 12, we see that

$$\text{Fix}(g^2) = \text{Fix}(g^4) = \text{Fix}(g^6) = \text{Fix}(g^{12}) = \{\text{two points}\}$$

and $\text{Fix}(g)$ is either empty or two points. We claim that $\text{ord}(g) = 5 \cdot 12$. Suppose that g^{30} is symplectic. Then by Proposition 2.5, $\text{Fix}(g^{30})$ consists of 8 points, on which g acts. The order of this action divides 30, so g^6 must fix at least 3 points. This proves that g^{30} is not symplectic. Suppose that g^{20} is symplectic. Then $\text{Fix}(g^{20})$ consists of 6 points, on which g acts with order dividing 20, so g^4 fixes 6 points. This proves that g^{20} is not symplectic. The claim is proved. Then $\zeta_{12} \in [g^{5*}]$, thus $[g^*]$ contains ζ_{12} or ζ_{60} . Checking all possible cases for $[g^*]$, we see that $e(\text{Fix}(g^{30})) = -16, -12, -8, 0$ or 16 and there are 13 possibilities for the pair $(e(\text{Fix}(g^{30})), e(\text{Fix}(g^{10})))$. In each case by considering the fixed loci of the tame automorphisms $\text{Fix}(g^{30})$, $\text{Fix}(g^{10})$, $\text{Fix}(g^5)$, $\text{Fix}(g^{15})$, one can show, using the inclusion relation among these sets, that $\text{Fix}(g^{12})$ is either empty or contains at least 3 points. \square

Lemma 9.9. *Let g be an automorphism of order $5n$, $5 \nmid n$. If g^n fixes exactly one point and $X/\langle g^n \rangle$ is a K3 surface with one rational double point, then $n = 1, 2, 3, 4, 6$.*

Proof. Let $u := g^n$. Then u has order 5 and by Proposition 9.1

$$[u^*] = [1, 1.13, (\zeta_5 : 4).2]$$

where the first eigenvalue corresponds to a g -invariant ample divisor.

Claim: $n \neq 2^3$.

Suppose that $n = 2^3$. Then $u = g^8$ and

$$\text{Fix}(g) = \text{Fix}(g^2) = \text{Fix}(g^4) = \text{Fix}(g^8) = \{\text{a point}\}.$$

By Theorem 1.4, ζ_8 must appear in $[g^*]$. The proof will be divided into 3 cases according to the possibility of $[g^*]$.

Case 1. $[g^*] = [1, (\zeta_8 : 4).3, \pm 1, \eta_1, \dots, \eta_8]$ where η_1, \dots, η_8 is a combination of $\zeta_{20} : 8$, $\zeta_{10} : 4$, $\zeta_5 : 4$. The tame involution $s = g^{20}$ has $e(X^s) = 0$ and $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 10$, hence is non-symplectic. Since $e(\text{Fix}(g^{10})) = -4$ or 12 , X^s is not empty. If X^s consists of elliptic curves,

then $\sum_{j=0}^2 (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(X^s, \mathbb{Q}_l)) = 0$, but $\sum_{j=0}^4 (-1)^j \text{Tr}(g^{28*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) = -10$ contradicting Deligne-Lusztig (Proposition 2.3). Thus X^s consists of a curve C_{d+1} of genus $d+1$ and d smooth rational curves, where $1 \leq d \leq 9$. The action of g^4 on X^s is of order 5 with exactly 1 fixed point. This is possible only if

- (1) $d = 5r$ and the d smooth rational curves form r orbits under g^4 ; or
- (2) $d = 5r + 1$, g^4 fixes one point of a smooth rational curve and the remaining $d - 1$ rational curves form r orbits under g^4 .

In case (1), $d = 5$ and the tame automorphism g^{10} preserves each of the 5 smooth rational curves. Thus $e(\text{Fix}(g^{10})) \geq 10$ or $e(\text{Fix}(g^{10})) = 2d + e(C_{d+1}) = 0$. Since the tame automorphism g^{10} has $e(\text{Fix}(g^{10})) = -4$ or 12 , we conclude that $e(\text{Fix}(g^{10})) = 12$ and g^{10} fixes two points on C_6 , each of which must be fixed by g^2 . But $\text{Fix}(g^2)$ is a point.

In case (2), $d = 1$ or 6 and g^4 acts freely on C_2 or C_7 . But no curve of genus 2 or 7 admits an order 5 free action in any characteristic.

Case 2. $[g^*] = [1, (\zeta_8 : 4).2, \tau_1, \dots, \tau_5, \eta_1, \dots, \eta_8]$ where τ_1, \dots, τ_5 is a combination of $\zeta_4 : 2, -1, 1$ and η_1, \dots, η_8 is a combination of $\zeta_{20} : 8, \zeta_{10} : 4, \zeta_5 : 4$. The tame involution $s = g^{20}$ has $e(X^s) = 8$ and $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 14$. If s is symplectic, then X^s consists of 8 points on which g acts. By considering all possible orbit decomposition of the 8 points under g , we infer that g^4 has at least 3 fixed points, a contradiction. Thus s is non-symplectic and X^s is either a union of elliptic curves and 4 smooth rational curves or a union of a curve C_{d-3} of genus $d-3$ and d smooth rational curves, where $3 \leq d \leq 13$. In the first case, the order 5 action of $u = g^8$ on $\text{Fix}(g^{20})$ preserves all the 4 smooth rational curves, hence fixes at least 4 points, a contradiction. Thus we have the second case. The action of g^4 on X^s is of order 5 with exactly 1 fixed point. This is possible only if

- (1) $d = 5r$ and the d smooth rational curves form r orbits under g^4 ; or
- (2) $d = 5r + 1$, g^4 fixes one point of a smooth rational curve and the remaining $d - 1$ rational curves form r orbits under g^4 .

In case (1), $d = 5$ or 10 and the tame automorphism g^{10} preserves each of the d smooth rational curves.

Assume that $d = 5$. Then $e(\text{Fix}(g^{10})) \geq 10$ or $e(\text{Fix}(g^{10})) = 10 + e(C_2) = 8$. From the possibilities of $[g^{10*}]$, we see that $e(\text{Fix}(g^{10})) = 8, 12$ or 16 . Suppose that $e(\text{Fix}(g^{10})) = 8$. Then $C_2 \subset \text{Fix}(g^{10})$, $[\tau_1, \dots, \tau_5] = [(\zeta_4 : 2).2, \pm 1]$ and $[\eta_1, \dots, \eta_8]$ is a combination of $\zeta_{10} : 4, \zeta_5 : 4$. Since g^5 preserves each of the 5 smooth rational curves, we see from $[g^{5*}]$ that $e(\text{Fix}(g^5)) = 10$ or 12 . In the first case, g^5 is a free involution of C_2 , a contradiction. In the second case, g^5 fixes 2 points on C_2 . Since g acts on the 2 points, g^2 fixes both, a contradiction. Suppose that $e(\text{Fix}(g^{10})) = 12$. Then g^{10} fixes 2 points on C_2 . Since g acts on the 2 points, g^2 fixes both, a contradiction. Suppose that $e(\text{Fix}(g^{10})) = 16$. Then g^{10} fixes 6 points on C_2 and g^5 is an

order 4 automorphism of C_2 , hence fixes 2 points. Then g acts on the 2 points and g^2 fixes both, a contradiction.

Assume that $d = 10$. Then $e(\text{Fix}(g^{10})) \geq 20$ or $e(\text{Fix}(g^{10})) = 20 + e(C_7) = 8$. From the possibilities of $[g^{10*}]$, we see that $e(\text{Fix}(g^{10})) = 8$. Then $C_2 \subset \text{Fix}(g^{10})$, $[\tau_1, \dots, \tau_5] = [(\zeta_4 : 2).2, \pm 1]$ and $[\eta_1, \dots, \eta_8]$ is a combination of $\zeta_{10} : 4$, $\zeta_5 : 4$. Suppose that $C_2 \subset \text{Fix}(g^5)$. Then since g^5 preserves either each of the 10 smooth rational curves or none, $e(\text{Fix}(g^5)) = 18$ or -2 . None of them is compatible with $[g^{5*}]$.

In case (2), $d = 6$ or 11 and g^4 acts freely on C_3 or C_8 . But no curve of genus 3 or 8 admits an order 5 free action in any characteristic.

Case 3. $[g^*] = [1, \zeta_8 : 4, \tau_1, \dots, \tau_9, \eta_1, \dots, \eta_8]$ where τ_1, \dots, τ_9 is a combination of $\zeta_4 : 2$, -1 , 1 and η_1, \dots, η_8 is a combination of $\zeta_{20} : 8$, $\zeta_{10} : 4$, $\zeta_5 : 4$. In this case, $s = g^{20}$ has $e(X^s) = 16$ and $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 18$, hence is non-symplectic. Thus X^s is either a union of elliptic curves and 8 smooth rational curves or a union of a curve C_{d-7} of genus $d - 7$ and d smooth rational curves, where $7 \leq d \leq 17$. In the first case, the order 5 action of $u = g^8$ on $\text{Fix}(g^{20})$ preserves at least 3 smooth rational curves, hence fixes at least 3 points, a contradiction. Thus we have the second case. Applying Deligne-Lusztig (Proposition 2.3) to $g^{28} = su$, we get

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(X^s, \mathbb{Q}_l)) = \sum_{j=0}^4 (-1)^j \text{Tr}(g^{28*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) = 6.$$

The action of $u = g^8$ on X^s is of order 5 with at most 1 fixed point. This is possible only if

- (1) $d = 5r$ and the d smooth rational curves form r orbits under $u = g^8$;
or
- (2) $d = 5r + 1$, $u = g^8$ fixes one point of a smooth rational curve and the remaining $d - 1$ rational curves form r orbits under u .

In case (1), $d = 10$ or 15 . If $d = 10$, then

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(X^s, \mathbb{Q}_l)) = \sum_{j=0}^2 (-1)^j \text{Tr}(u^* | H_{\text{et}}^j(C_3, \mathbb{Q}_l)) \leq 1.$$

If $d = 15$, then Deligne-Lusztig does not work as it may happen that $\text{Tr}(u^* | H_{\text{et}}^1(C_8, \mathbb{Q}_l)) = -4$. We need a different argument. The sum of the five (-2) -curves in each orbit gives a u -invariant divisor V_i and the four divisors C_8, V_1, V_2, V_3 generate a rank 4 invariant sublattice of $\text{NS}(X)$. This sublattice represents 0, e.g.,

$$(5C_8 + 5V_1 + 3V_2 + V_3)^2 = 0.$$

Thus X admits a u -invariant elliptic fibration, contradicting Proposition 9.1. In case (2), $d = 11$ or 16 and $u = g^8$ acts freely on C_4 or C_9 . But no curve of genus 4 or 9 admits an order 5 free action in any characteristic.

Claim: $n \neq 3^2$.

Suppose that $n = 3^2$. Then $u = g^9$. Let

$$s = g^{15}.$$

By Theorem 1.4, ζ_9 must appear in $[g^*]$. Suppose that $\zeta_9 : 6$ appears only once in $[g^*]$, i.e.,

$$[g^*] = [1, \zeta_9 : 6, \eta_1, \dots, \eta_7, \tau_1, \dots, \tau_8]$$

where $\eta_j = 1, \zeta_3$ or ζ_3^2 , and $[\tau_1, \dots, \tau_8] = [\zeta_{15} : 8]$ or $[(\zeta_5 : 4).2]$. Then

$$[g^{15*}] = [1, (\zeta_3 : 2).3, 1.7, 1.8],$$

hence the order 3 tame automorphism $s = g^{15}$ is non-symplectic by Lemma 2.6 and the quotient $X/\langle s \rangle$ is rational. Then by Proposition 2.2,

$$\text{rank NS}(X)^s = \dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 16.$$

It follows that η_1, \dots, η_7 are supported on $\text{NS}(X)$ and

$$\text{rank NS}(X)^u \geq 8.$$

Since a hyperbolic lattice of rank ≥ 5 represents 0, so does $\text{NS}(X)^u$. Then X admits a u -invariant elliptic fibration, contradicting Proposition 9.1. Suppose now that

$$[g^*] = [1, (\zeta_9 : 6).2, 1, \tau_1, \dots, \tau_8]$$

where $[\tau_1, \dots, \tau_8] = [\zeta_{15} : 8]$ or $[(\zeta_5 : 4).2]$. By computing $[g^{15*}]$ we see that $e(X^s) = 6$. Note that

$$\text{Fix}(g) = \text{Fix}(g^3) = \text{Fix}(g^9) = \{\text{a point}\}.$$

The order 5 action of $u = g^9$ on X^s has exactly one fixed point.

Assume that $u = g^9$ fixes an isolated point of X^s . Then X^s consists of $5t + 1$ points, $5d$ smooth rational curves, a curve C_r of genus r , and possibly some elliptic curves if $r \leq 1$. Since u acts freely on C_r , $2r - 2 = 5(2r' - 2)$ where r' is the genus of $C_r/\langle u \rangle$. Thus $u^*|H_{\text{et}}^1(C_r, \mathbb{Q}_l)$ has $[u^*] = [1.(2r'), (\zeta_5 : 4).(2r' - 2)]$, hence

$$\text{Tr}(u^*|H_{\text{et}}^1(C_r, \mathbb{Q}_l)) = 2r' - (2r' - 2) = 2.$$

Then

$$\sum_{j=0}^2 (-1)^j \text{Tr}(u^*|H_{\text{et}}^j(X^s, \mathbb{Q}_l)) = 1 + \sum_{j=0}^2 (-1)^j \text{Tr}(u^*|H_{\text{et}}^j(C_r, \mathbb{Q}_l)) = 1.$$

But

$$\sum (-1)^j \text{Tr}(g^{24*}|H_{\text{et}}^j(X, \mathbb{Q}_l)) = -4,$$

contradicting Deligne-Lusztig.

Assume that $u = g^9$ fixes a point of a smooth rational curve in X^s . Then X^s consists of $5t$ points, $5d + 1$ smooth rational curves, a curve C_r of genus r , and possibly some elliptic curves if $r \leq 1$. Then $e(X^s) = 5t + 10d + 2 + 2 - 2r = 6$. Since u acts freely on C_r , $2 - 2r \equiv 0 \pmod{5}$, thus $2 \equiv 6 \pmod{5}$, a contradiction.

Assume that $u = g^9$ fixes a point of a curve C_r of genus r in X^s , and that X^s consists of $5t$ points, $5d$ smooth rational curves, the curve C_r , and possibly some elliptic curves if $r \leq 1$. Since $e(X^s) = 5t + 10d + 2 - 2r = 6$, $t = 2t'$ and $r = 5t' + 5d - 2 \geq 3$. Unfortunately, Deligne-Lusztig does not work in this case, as $u^*|H_{\text{et}}^1(C_r, \mathbb{Q}_l)$ may have $[u^*] = [1, (2t' + 2d + 4), (\zeta_5 : 4), (2t' + 2d - 2)]$. The action of g on the $10t'$ isolated points of X^s has orbits of length 15 or 5 only, since $u = g^9$ has orbits of length 5 only. We see that g^5 fixes no point in any orbit of g of length 15, and fixes each point in any orbit of length 5. Note that $d \leq 1$, hence g^5 preserves each of the $5d$ smooth rational curves. Now we claim that $C_r \not\subseteq \text{Fix}(g^5)$. Otherwise,

$$e(\text{Fix}(g^5)) = e(\text{Fix}(g^{15})) - 15a = 6 - 15a$$

for some non-negative integer a , which contradicts

$$\sum (-1)^j \text{Tr}(g^{5*}|H_{\text{et}}^j(X, \mathbb{Q}_l)) = 0 \text{ or } 12.$$

This proves the claim and g^5 acts non-trivially on C_r . Since g has a fixed point on C_r , so does g^5 . Thus $e(\text{Fix}(g^5)) > 0$, hence $e(\text{Fix}(g^5)) = 12$. From this we infer that g^5 has 12, 7 or 2 fixed points on C_r . In any case, g has at least 2 fixed points.

Claim: n cannot be a prime ≥ 17 .

Suppose that n is a prime ≥ 17 . Then by Theorem 1.4, either ζ_n or ζ_{5n} must appear in $[g^*]$. But $\phi(n) > 13$ and $\phi(45) > 8$. This proves the claim.

Claim: $n \neq 11$.

Suppose that $n = 11$. Then by Theorem 1.4, ζ_{11} must appear in $[g^*]$ and

$$[g^*] = [1, \zeta_{11} : 10, 1.3, (\zeta_5 : 4).2].$$

The order 11 automorphism $s = g^5$ is non-symplectic and has

$$[s^*] = [g^{5*}] = [1, \zeta_{11} : 10, 1.3, 1.8].$$

Note that $e(\text{Fix}(g^5)) = 13$. Since $\text{Fix}(g) = \text{Fix}(g^{11})$, the order 5 action of $u = g^{11}$ on X^s has exactly one fixed point.

Assume that $u = g^{11}$ fixes an isolated point of X^s . Then X^s consists of $5t + 1$ points, $5d$ smooth rational curves, a curve C_r of genus r , and possibly some elliptic curves if $r \leq 1$. Then $e(X^s) = 5t + 1 + 10d + 2 - 2r = 13$. Since u acts freely on C_r , $2 - 2r \equiv 0 \pmod{5}$, thus $1 \equiv 13 \pmod{5}$, a contradiction.

Assume that $u = g^{11}$ fixes a point of a smooth rational curve in X^s . Then X^s consists of $5t$ points, $5d + 1$ smooth rational curves, a curve C_r of genus r , and possibly some elliptic curves if $r \leq 1$. Then $e(X^s) = 5t + 10d + 2 + 2 - 2r = 13$. Since u acts freely on C_r , $2 \equiv 13 \pmod{5}$, a contradiction.

Assume that $u = g^{11}$ fixes a point of a curve C_r of genus r in X^s , and that X^s consists of $5t$ points, $5d$ smooth rational curves R_1, \dots, R_{5d} , the curve C_r , and possibly some elliptic curves if $r \leq 1$. Since $e(X^s) = 5t + 10d + 2 - 2r = 13$,

$$2r - 2 = 5t + 10d - 13 \geq 2.$$

Deligne-Lusztig does not work in this case, as $u^*[H_{\text{et}}^1(C_r, \mathbb{Q}_l)]$ may have $[u^*] = [1.(t+2d-3), (\zeta_5 : 4).(t+2d-2)]$. Since $s = g^5$ is non-symplectic, we may assume that $s^*\omega_X = \zeta_{11}^5\omega_X$. An isolated fixed point of g^5 is one of the following 5 types: $\frac{1}{11}(1, 4)$, $\frac{1}{11}(2, 3)$, $\frac{1}{11}(6, 10)$, $\frac{1}{11}(7, 9)$, $\frac{1}{11}(8, 8)$. Let $5t_i$ be the number of isolated fixed point of the i -th type. Then

$$\sum t_i = t.$$

The quotient $X' = X/\langle g^5 \rangle$ is a rational surface with

$$K_{X'} = -\frac{10}{11}\left(C'_r + \sum_{i=1}^{5d} R'_i\right)$$

where C'_r and R'_i are the images of C_r and R_i . Let $\varepsilon : Y \rightarrow X'$ be a minimal resolution. Then

$$K_Y = \varepsilon^*K_{X'} - \sum D_p$$

where D_p is an effective \mathbb{Q} -divisor supported on the exceptional set of the singular point $p \in X'$. Thus

$$K_Y^2 = K_{X'}^2 + \sum D_p^2 = K_{X'}^2 - \sum K_Y D_p.$$

See, e.g., Lemma 3.6 [9] for the formulas of D_p and $K_Y D_p$, which are valid for tame quotient singular points in positive characteristic. We compute

$$K_Y^2 = 10 - \rho(Y) = 10 - \{\rho(X') + 10t_1 + 20t_2 + 25t_3 + 10t_4 + 5t_5\}.$$

Since $C_r'^2 = 11(5t + 10d - 13)$ and $R_i'^2 = -22$, we have

$$K_{X'}^2 = \frac{10^2}{11^2}\left(11(5t + 10d - 13) - 22 \cdot 5d\right) = \frac{10^2}{11}\left(5 \sum t_i - 13\right).$$

Since $K_Y D_p = \frac{20}{11}, \frac{6}{11}, \frac{5}{11}, \frac{32}{11}, \frac{81}{11}$ for p of each type and $\rho(X') = \dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 12$, we finally have

$$510t_1 + 690t_2 + 750t_3 + 450t_4 + 150t_5 = 1278.$$

The equation has no solution for (t_1, \dots, t_5) .

Claim: $n \neq 13$.

We omit the proof, which is quite similar to the previous case.

Claim: $n \neq 7$.

Suppose that $n = 7$. Then $u = g^7$. Let $s = g^5$. By Theorem 1.4, ζ_7 must appear in $[g^*]$. Suppose that

$$[g^*] = [1, \zeta_7 : 6, 1.7, (\zeta_5 : 4).2].$$

Then $[g^{5*}] = [1, \zeta_7 : 6, 1.7, 1.8]$, hence the order 7 automorphism $s = g^5$ is non-symplectic by Lemma 2.6 and the quotient $X/\langle s \rangle$ is rational. Then by Proposition 2.2, $\text{rank NS}(X)^s = \dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 16$. It follows that 1.7 in $[g^*]$ are supported on $\text{NS}(X)$ and $\text{rank NS}(X)^u \geq 8$. Thus X admits a u -invariant elliptic fibration, contradicting Proposition 9.1. Suppose that

$$[g^*] = [1, (\zeta_7 : 6).2, 1, (\zeta_5 : 4).2].$$

By computing $[g^{5*}]$ we see that

$$e(X^s) = e(\text{Fix}(g^5)) = 10.$$

Since $\text{Fix}(g) = \text{Fix}(g^7)$, the order 5 action of $u = g^7$ on X^s has exactly one fixed point.

Assume that $u = g^7$ fixes an isolated point of X^s . Then X^s consists of $5t + 1$ points, $5d$ smooth rational curves, a curve C_r of genus r , and possibly some elliptic curves if $r \leq 1$. Then $e(X^s) = 5t + 1 + 10d + 2 - 2r = 10$. Since u acts freely on C_r , $2 - 2r \equiv 0 \pmod{5}$, thus $1 \equiv 0 \pmod{5}$, a contradiction.

Assume that $u = g^7$ fixes a point of a smooth rational curve in X^s . Then X^s consists of $5t$ points, $5d + 1$ smooth rational curves, a curve C_r of genus r , and possibly some elliptic curves if $r \leq 1$. Then $e(X^s) = 5t + 10d + 2 + 2 - 2r = 10$. Since u acts freely on C_r , $2 \equiv 0 \pmod{5}$, a contradiction.

Assume that $u = g^7$ fixes a point of a curve C_r of genus r in X^s , and that X^s consists of $5t$ points, $5d$ smooth rational curves, the curve C_r , and possibly some elliptic curves if $r \leq 1$. In this case, Deligne-Lusztig does not work. The argument using the rational quotient $X' = X/\langle g^5 \rangle$ as in Claim $n \neq 11$ does not work, either; indeed, there are two solutions for (t_1, t_2, t_3) where $5t_1, 5t_2, 5t_3$ are the numbers of isolated fixed points respectively of type $\frac{1}{7}(1, 4), \frac{1}{7}(2, 3), \frac{1}{7}(6, 6)$. We need a new argument. Let g' be the automorphism induced by g on the quotient $X/\langle g^7 \rangle$. It is of order 7, acts on the minimal resolution, hence on the 8 exceptional curves of type E_8 . It fixes the central component point-wisely. By analyzing the action of g' at the intersection point of two components we infer that the image C'_r cannot meet any of the 8 components, contradicting that g has a fixed point on C_r .

Claim: $n \neq 12$.

Suppose that $n = 12$. Then $u = g^{12}$ and

$$\text{Fix}(g) = \text{Fix}(g^2) = \text{Fix}(g^3) = \text{Fix}(g^4) = \text{Fix}(g^6) = \text{Fix}(g^{12}) = \{\text{one point}\}.$$

As in Lemma 9.8, g^{30} is non-symplectic, thus $\text{ord}(g) = 5.12$ or 15.4 .

Assume that $\zeta_{12} \in [g^*]$. Checking all possible cases for $[g^*]$, we see that $e(\text{Fix}(g^{30})) = -16, -12, -8, -4, 0, 4, 8, 12$ or 16 and there are 57 possibilities for the pair $(e(\text{Fix}(g^{30})), e(\text{Fix}(g^{10})))$. In each case by considering the fixed loci of the tame automorphisms $\text{Fix}(g^{30}), \text{Fix}(g^{10}), \text{Fix}(g^5), \text{Fix}(g^{15})$, one can show, using the inclusion relation among these sets, that $\text{Fix}(g^{12})$ is either empty or contains at least 2 points, except one case. In the last case, the quotient $X/\langle g^{20} \rangle$ can be shown to be rational and have Picard number which is not compatible with the numerical invariants of the minimal resolution (see Claim $n \neq 11$).

Assume that $\zeta_{12} \notin [g^*]$. Since $\zeta_{60} \notin [g^*]$, $\text{ord}(g) = 15.4$ and g^{20} is symplectic. Thus $[g^{20*}] = [1, 1.9, (\zeta_3 : 2).6]$ and $\zeta_4 \in [g^{15*}]$. The latter implies that ζ_4 or ζ_{12} or $\zeta_{20} \in [g^*]$. By assumption $\zeta_{12} \notin [g^*]$. Since $\text{Fix}(g)$ is a point and $\text{Fix}(g) \subset \text{Fix}(g^{20})$, g acts on the 6 point set $\text{Fix}(g^{20})$ fixing one point

and rotating five. Thus $\text{Fix}(g^{10}) = \text{Fix}(g^{20})$. In particular, $e(\text{Fix}(g^{10})) = 6$. If $\zeta_{20} \in [g^*]$, then

$$[g^*] = [1, \zeta_{20} : 8, \pm 1, \pm \zeta_3 : 2, \pm \zeta_3 : 2, \pm \zeta_3 : 2, \pm \zeta_3 : 2, \pm \zeta_3 : 2, \pm \zeta_3 : 2]$$

where $\pm \zeta_3 : 2$ means $\zeta_3 : 2$ or $\zeta_6 : 2$. In any case, $[g^{10*}] = [1, -1.8, 1, (\zeta_3 : 2).6]$ and $e(\text{Fix}(g^{10})) = -10 \neq 6$. If $\zeta_{20} \notin [g^*]$, then $\zeta_4 \in [g^*]$. Since $[g^{20*}] = [1, 1.9, (\zeta_3 : 2).6]$, we infer that

$$[g^*] = [1, \zeta_4 : 2, \pm \zeta_3 : 2, \pm \zeta_3 : 2, \eta_1, \dots, \eta_7, \pm \zeta_{15} : 8]$$

where η_1, \dots, η_7 is a combination of $\zeta_4 : 2, -1, 1$. In each case, we compute $e(\text{Fix}(g^{10})) = 2, -2, -6, -10$, none is equal to 6. \square

Lemma 9.10. *Let g be an automorphism of order $5n$, $5 \nmid n$. If g^n fixes exactly one point and $X/\langle g^n \rangle$ is rational, then $n = 1, 2, 3, 4, 6$.*

Proof. Let $u := g^n$. Then u has order 5 and by Proposition 9.1

$$[u^*] = [1, 1.5, (\zeta_5 : 4).4]$$

where the first eigenvalue corresponds to a g -invariant ample divisor.

Claim: $n \neq 3^2$, n cannot be a prime ≥ 7 .

The proof is the same as in Case 1, Lemma 9.7.

Claim: $n \neq 2^3$.

Suppose that $n = 2^3$. Then $\text{Fix}(g) = \text{Fix}(g^2) = \text{Fix}(g^4) = \text{Fix}(g^8) = \{\text{a point}\}$. The proof will be divided into 3 cases.

Case 1. $[g^*] = [1, \zeta_8 : 4, \pm 1, \zeta_{40} : 16]$.

The tame involution $s = g^{20}$ has $e(X^s) = -16$ and $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^s = 2$, hence is non-symplectic. Thus X^s consists of either a curve C_9 of genus 9 or a smooth rational curve R and a curve C_{10} of genus 10. In the second case, the order 5 action of $u = g^8$ fixes one point on R and at least one on C_{10} . Thus $X^s = C_9$. Since $e(\text{Fix}(g^{10})) = 4$ and $\text{Fix}(g^{10}) \subset X^s$, we infer that $\text{Fix}(g^{10})$ consists of 4 points on C_9 . Since g acts on these 4 points, g^2 fixes all of them, a contradiction.

Case 2. $[g^*] = [1, \eta_1, \dots, \eta_5, \zeta_{40} : 16]$ where η_1, \dots, η_5 is a combination of $\zeta_4 : 2, -1, 1$. In this case, the proof is the same as that of Case 2, Claim: $n \neq 2^3$, Lemma 9.8.

Case 3. $[g^*] = [1, \zeta_8 : 4, \pm 1, \eta_1, \dots, \eta_{16}]$ where η_1, \dots, η_{16} is a combination of $\zeta_{20} : 8, \zeta_{10} : 4, \zeta_5 : 4$. In this case, the proof is just a copy of Case 3, Claim: $n \neq 2^3$, Lemma 9.8, except that the action of g^4 on X^s is of order 5 with exactly 1 fixed point.

Claim: $n \neq 12$.

Suppose that $n = 12$. Then $u = g^{12}$ and

$$\text{Fix}(g) = \text{Fix}(g^2) = \text{Fix}(g^3) = \text{Fix}(g^4) = \text{Fix}(g^6) = \text{Fix}(g^{12}) = \{\text{one point}\}.$$

As in Lemma 9.8, g^{30} is not symplectic, thus $\text{ord}(g) = 5.12$ or 15.4 .

Assume that $\zeta_{12} \in [g^*]$ or $\zeta_{60} \in [g^*]$. Checking all possible cases for $[g^*]$, we see that $e(\text{Fix}(g^{30})) = -16, -12, -8, 0$, or 16 and there are 13 possibilities for the pair $(e(\text{Fix}(g^{30})), e(\text{Fix}(g^{10})))$. Each case can be ruled out in the same way as in Lemma 9.8.

Assume that $\zeta_{12}, \zeta_{60} \notin [g^*]$. Then $\text{ord}(g) = 15.4$ and g^{20} is symplectic. Thus $[g^{20*}] = [1, 1.9, (\zeta_3 : 2).6]$ and $\zeta_4 \in [g^{15*}]$. The latter implies that ζ_4 or ζ_{12} or ζ_{20} or $\zeta_{60} \in [g^*]$. By assumption, ζ_4 or $\zeta_{20} \in [g^*]$. Since $\text{Fix}(g)$ is a point, g acts on the 6 point set $\text{Fix}(g^{20})$ fixing one point and rotating five. Thus $\text{Fix}(g^{10}) = \text{Fix}(g^{20})$. In particular, $e(\text{Fix}(g^{10})) = 6$. If $\zeta_4 \in [g^*]$, then it is easy to infer that $[g^{20*}] \neq [1, 1.9, (\zeta_3 : 2).6]$. If $\zeta_{20} \in [g^*]$, then

$$[g^*] = [1, \zeta_{20} : 8, \pm 1, \pm \zeta_{15} : 8, \pm \zeta_3 : 2, \pm \zeta_3 : 2]$$

where $\pm \zeta_{15} : 8$ means $\zeta_{15} : 8$ or $\zeta_{30} : 8$. In any case, $[g^{10*}] = [1, -1.8, 1, (\zeta_3 : 2).6]$ and $e(\text{Fix}(g^{10})) = -10 \neq 6$. \square

Example 9.11. In char $p = 5$, there are K3 surfaces with an automorphism of order 30 or 20.

$$(1) X_{30} : y^2 = x^3 + (t^5 - t)^2, g_{30}(t, x, y) = (t + 1, \zeta_3 x, -y);$$

$$(2) X_{20} : y^2 = x^3 + (t^5 - t)x, g_{20}(t, x, y) = (t + 1, -x, \zeta_4 y).$$

The surface X_{30} has 6 *IV*-fibres at $t = \infty, t^5 - t = 0$; X_{20} has a *III**-fibre at $t = \infty$ and 5 *III*-fibres at $t^5 - t = 0$ ([6], 5.8).

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